Online Appendix for "Advising the Management: A Theory of Shareholder Engagement"

The Online Appendix presents the analysis of the extensions of the model.

A.1 General model

In this section, we consider a more general version of the model, in which different shareholders get signals of different quality, and the manager is privately informed as well. Specifically, there is a set of investors indexed by i (who observe signals $\theta_1, ..., \theta_N$), and the manager, indexed by m, who observes signal θ_{N+1}. The payoffs of each investor and the manager are as in the basic model. The state is:

$$Z = \sum_{i=1}^{N+1} c_i \theta_i, \tag{56}$$

where $c_i > 0$. Coefficients c_i can take any positive values and do not need to sum up to one. An agent with a higher c_i can be interpreted as being more informed. Thus, for simplicity, we focus on the case of no residual uncertainty, i.e., the state is perfectly known to all investors and the manager as a whole, but the model can be easily generalized further, to capture residual uncertainty.

As in the basic model, θ_i is a binary signal equal to 1 with probability φ and 0 with probability $1-\varphi$, and agents may potentially disagree about φ: agent i's prior of φ is characterized by the Beta distribution with parameters $(\rho_i, \tau - \rho_i)$. We allow for any general set of investor beliefs, ρ_i, as well as belief ρ_m of the manager. The rest of the assumptions (e.g., the timing and the trading stage) are exactly as in the basic model. In what follows, we present the analogs of the core results in the main model for this more general setup.

Lemma OA.1 (Optimal action of the manager). *Suppose that after the communication stage, the manager knows subset R of signals. Then his optimal action is*

$$a_m(\theta_R) = b + \sum_{i \in R} c_i \theta_i + \frac{\rho_m + \sum_{i \in R} \theta_i}{\tau + |R|} \sum_{j \in -R} c_j, \tag{57}$$

where $|R|$ is the number of signals in R.

Proposition OA.1 (IC constraint for truthful reporting). *Suppose that the manager learns subset R_i of signals (which includes his own signal θ_m but not θ_i) and does not know all the other signals, $-R_i$. Then shareholder i reports his signal truthfully if and only if*

$$2\left|b + \frac{\sum_{j \in -R_i \setminus \{i\}} c_j}{\tau + |R_i| + 1}(\rho_m - \rho_i)\right| \le c_i + \frac{\sum_{j \in -R_i \setminus \{i\}} c_j}{\tau + |R_i| + 1}. \tag{58}$$

Condition (58) is the analog of (11) in the main model. Note that regardless of the source of communication frictions, shareholder i is more likely to report his signal truthfully if his information is more important: the IC constraint (58) is relaxed when c_i increases. Intuitively, the shareholder faces the same trade-off as described in the paper: while he wants to tilt the manager in the direction of his optimal action (the benefit of misreporting), he is also afraid to tilt it too much, away even from his own optimal action, i.e., to "overshoot" (the cost of misreporting). As the agent's information becomes more important and hence the manager is expected to react more strongly to the agent's message (as captured by the term c_i on the right-hand side), this fear makes the agent more reluctant to misreport.

To show that the existence of the complementarity and substitution effects in communication extends to this more general model, consider two extreme cases:

Case 1. $b = 0$ **but heterogeneous beliefs**

In this case, shareholder i reports his signal truthfully if and only if[27]

$$|\rho_m - \rho_i| \leq \frac{1}{2}\left[1 + c_i \frac{\tau + |R_i| + 1}{\sum_{j\in -R_i\setminus\{i\}} c_j}\right]. \tag{59}$$

Hence, as in the basic model, shareholders' communication decisions are complements: the more information the manager gets from other shareholders (i.e., the higher is $|R_i|$ and the lower is $\sum_{j\in -R_i\setminus\{i\}} c_j$), the more likely it is that shareholder i will also truthfully communicate his information.

Case 2. $b > 0$ **and homogenous beliefs**

In this case, $\rho_i = \rho_m = \rho$, so shareholder i reports his signal truthfully if and only if

$$b \leq \frac{1}{2}\left[c_i + \frac{\sum_{j\in -R_i\setminus\{i\}} c_j}{\tau + |R_i| + 1}\right]. \tag{60}$$

Hence, as in the basic model, shareholders' communication decisions are substitutes: the more information the manager gets from other shareholders (i.e., the higher is $|R_i|$ and the lower is $\sum_{j\in -R_i\setminus\{i\}} c_j$), the less likely it is that shareholder i will truthfully communicate his information.

Next, we derive each investor's valuation of the shares as a function of the information $|R|$ that the manager is expected to have at the decision-making stage.

Lemma OA.2 (Ex-ante payoffs). *Suppose that in equilibrium, the manager learns subset R of the signals and does not learn all the other signals, $-R$. Then agent i's valuation of each share is given by:*

$$\mathbb{E}_i[U_i|R] = u_0 - b^2 - \mathsf{A}_{im}(R) - \mathsf{B}_i(R) - \mathsf{C}_{im}(R), \tag{61}$$

[27] Here if $-R\setminus\{i\}$ is an empty set, the right-hand side of (18) is equal to infinity, i.e., (18) is always satisfied.

where

$$\begin{aligned}
\mathsf{A}_{im}(R) &= \tfrac{2b(\rho_m-\rho_i)}{\tau+|R|}\sum_{j\in -R} c_j,\\
\mathsf{B}_i(R) &= \tfrac{\rho_i(\tau-\rho_i)}{\tau(\tau+1)}\left(\sum_{j\in -R} c_j^2 + \tfrac{\left[\sum_{j\in -R} c_j\right]^2}{\tau+|R|}\right),\\
\mathsf{C}_{im}(R) &= \left[\tfrac{\rho_m-\rho_i}{\tau+|R|}\sum_{j\in -R} c_j\right]^2.
\end{aligned} \tag{62}$$

where $B_i(R), C_{im}(R)$ *are decreasing in* $|R|$ *and increasing in any* $c_j,\ j \in -R.$

It follows that as long as b is not too large, each agent's valuation is increasing in $|R|$, so the most informative communication equilibrium is Pareto efficient. Given that the complementarity and substitution properties continue to hold, it is easy to generalize the results of the paper to this setting.

A.2 A more general specification of differences in beliefs

In this section, we generalize the setting of Section A.1 of the Online Appendix even further: we assume that agent i has a prior belief that φ is distributed according to the Beta distribution with parameters $(\rho_i, \tau_i - \rho_i)$, i.e., we allow for heterogeneous τ_i across agents. We derive the incentive compatibility constraint for this specification and show that the complementarity in communication decisions extends to this specification.

Proposition OA.2. *Suppose that the manager learns subset* R_i *of signals (which includes his own signal* θ_m *but not* θ_i*) and does not know all the other signals,* $-R_i$*. Then shareholder* i *reports his signal truthfully if and only if*

$$\begin{aligned}
&\left| b\frac{(\tau_i+1)(\tau_m+|R_i|+1)}{\sum_{j\in -R_i\setminus\{i\}} c_j} + \frac{\tau_i-\tau_m}{2} + \rho_m(\tau_i+1) - \rho_i(\tau_m+1)\right|\\
\le\ & \frac{(\tau_i+1)\,c_i\,(\tau_m+|R_i|+1)}{2\sum_{j\in -R_i\setminus\{i\}} c_j} + \frac{\tau_i+1}{2} - \frac{\tau_i-\tau_m}{2}.
\end{aligned} \tag{63}$$

In particular, when $b=0$*, it reduces to*

$$\left|\frac{\tau_i-\tau_m}{2} + \rho_m(\tau_i+1) - \rho_i(\tau_m+1)\right| \le \frac{(\tau_i+1)\,c_i\,(\tau_m+|R_i|+1)}{2\sum_{j\in -R_i\setminus\{i\}} c_j} + \frac{\tau_i+1}{2} - \frac{\tau_i-\tau_m}{2}. \tag{64}$$

This inequality is relaxed as R_i *expands and is always satisfied if* R_i *includes all signals other than* θ_i.

Since the right-hand side of (64) increases as R_i expands, shareholders' decisions are complements as in the basic model: more information revealed to the manager by some shareholders encourages other shareholders to report their information truthfully.

A.3 Costly information acquisition

Suppose that shareholders are not endowed with information: instead, shareholder i can incur cost $\kappa \geq 0$ to privately observe signal θ_i, and is uncertain about other signals. The timeline is as follows. After the trading stage, all shareholders of the firm simultaneously decide whether to incur a private cost to acquire their private signals. We assume that shareholders' information acquisition decisions are observed, and this happens after the communication stage.[28] Next, all shareholders simultaneously communicate their information to the manager, and the manager takes the action that maximizes his payoff. We look for equilibria in pure strategies at the information acquisition and communication stages. For simplicity, we focus on the case $b = 0$.

Proposition OA.3 (Number of shareholders and information acquisition). *Suppose $b = 0$, so that all N investors become shareholders. Then all shareholders find it optimal to acquire information if and only if $N \leq \hat{N}(\kappa)$, where $\hat{N}(\kappa)$ decreases in κ.*

Intuitively, each shareholder's incentives to acquire information decrease in N for two complementary reasons. First, the larger is the number of shareholders, the lower is each individual shareholder's stake, and hence the stronger is the free-riding effect: the shareholder bears the full cost κ but only captures a small fraction of the benefit. Second, the larger is the number of shareholders, the larger is the aggregate information that the shareholders possess, and hence, the lower is the marginal value of any additional signal.

Thus, the requirement that shareholders must pay the information acquisition cost imposes an upper bound on the number of shareholders who can communicate their views to the manager: $N \leq \hat{N}(\kappa)$. In particular, if the shareholder base is too dispersed, an equilibrium with information acquisition and communication by all shareholders does not exist. On the other hand, the fact that shareholders' communication decisions are complements imposes a lower bound on the number of shareholders who need to communicate with the manager in order for it to be incentive compatible for them to tell the truth ($N \geq K - \frac{\rho + K/2}{\Delta}$ from Proposition 2), i.e., ownership cannot be too concentrated either.

A.4 Comparative statics

Consider investor i, who believes that φ is distributed according to the Beta distribution with parameters $(\rho_i, \tau - \rho_i)$. The proof of Auxiliary Lemma A.1 in the appendix shows that from

[28] Without this assumption, a shareholder who deviates from his equilibrium strategy and does not invest in information, may want to mislead the manager and try to send a signal that he did not in fact acquire. Making the above assumption makes such deviations impossible and hence simplifies the analysis. In addition, assuming that information acquisition is observed after the communication stage rather than before simplifies the incentive compatibility constraint on information acquisition, because it implies that other shareholders do not change their behavior when one shareholder deviates to not acquiring information. However, most of the analysis would remain unchanged if information acquisition decisions were unobserved: the only difference would be an additional incentive compatibility constraint on information acquisition, which would not change the results qualitatively.

the perspective of investor i,

$$\mathbb{E}_i[\varphi] = \frac{\rho_i}{\tau}$$

and $\mathbb{E}[\varphi^2] = \frac{\rho_i(\rho_i+1)}{\tau(\tau+1)}$, and hence

$$Var_i[\varphi] = \frac{\rho_i}{\tau}\left(\frac{\rho_i+1}{\tau+1} - \frac{\rho_i}{\tau}\right) = \frac{\rho_i}{\tau}\frac{\tau-\rho_i}{\tau(\tau+1)}.$$

We are interested how the IC constraints for truthful communication are affected by the uncertainty about φ. Note that parameters τ and ρ_i affect both the mean and the variance of the distribution. We therefore would like to perform comparative statics in $Var_i[\varphi]$, while keeping the mean of the distribution fixed. To do this, suppose that we fix each investor's prior expectation of φ: denote $E_i \equiv \mathbb{E}_i[\varphi] = \frac{\rho_i}{\tau}$. Then

$$Var_i[\varphi] = E_i\frac{\tau - \tau E_i}{\tau(\tau+1)} = E_i\frac{1-E_i}{\tau+1}, \tag{65}$$

and the IC constraint (10) becomes

$$|(\tau + |R_i| + 1)\, b + (K - |R_i| - 1)\, \tau(E_m - E_i)| \leq \frac{\tau+K}{2}. \tag{66}$$

Hence, suppose we decrease τ but also simultaneously decrease all ρ_i and ρ_m proportionally to τ, in order to keep fixed the expectations of φ across investors and the manager: $E_i = \frac{\rho_i}{\tau}$ and $E_m = \frac{\rho_m}{\tau}$. Then (65) implies that this change in parameters corresponds to an increase in the variance of φ from each agent's perspective. How does the IC constraint (66) change as we make this parameter change? We consider two cases:

If $b = 0$, (66) is equivalent to

$$|(K - |R_i| - 1)\,(E_m - E_i)| \leq \frac{1}{2} + \frac{K}{2\tau}, \tag{67}$$

and it becomes more lax as τ decreases. Hence, under heterogeneous beliefs, truthful communication is more likely when there is more uncertainty.

If $\rho_i = \rho_m$ for all i, (66) is equivalent to

$$b \leq \frac{1}{2}\frac{\tau+K}{\tau+|R_i|+1}, \tag{68}$$

which also becomes more lax as τ decreases. Hence, under heterogeneous preferences, truthful communication is also more likely when there is more uncertainty.

To see the intuition, note that the manager's reaction to the shareholder's advice, i.e., by how much the manager's action changes if the shareholder misreports his signal θ_i, is given by $1 + \frac{K-|R_i|-1}{\tau+|R_i|+1}$. Hence, as τ decreases, and thus the variance of φ increases, the manager reacts more strongly to the shareholder's advice, because high variance means relatively uninformative priors. This makes misreporting more costly and truthful communication more likely.

A.5 Heterogeneous interpretation of information

In this section, we extend the model to capture different interpretations of signals by the shareholders and the manager. In particular, we follow models of differences of opinion in which agents disagree about the precision of signals, and each agent's belief about the precision of his own signal is higher than other agents' beliefs about it (e.g., Banerjee et al., 2009; Kyle et al., 2018).

Specifically, consider the setting of Section A.1: there is a set of investors indexed by i (who observe signals $\theta_1, ..., \theta_N$) and the manager, indexed by m, who observes signal θ_{N+1}. Assume, for simplicity, that $c_i = 1$ for all i in (56), as in the basic model. Also as in the basic model, agent i's prior is that φ is drawn from the Beta distribution with parameters $(\rho_i, \tau - \rho_i)$. The key difference from the main model is that each agent overestimates the importance of his own signal: agent i believes that the state of the world is equal to

$$Z_i = \gamma\theta_i + \sum_{j \neq i} \theta_j, \tag{69}$$

where $\gamma \geq 1$ captures the extent of heterogeneous interpretation of signals. In this setting, even if the agents knew all the signals that comprise the state (assume, for simplicity, that this is indeed the case, $K = N + 1$), they would still disagree about the state and thus the optimal action. Because our goal is to show that the complementarity effect in communication arises in this case as well, we assume $b = 0$ for ease of exposition.

The next result presents the constraint for truthful communication by shareholder i:

Proposition OA.4. *Suppose that the manager learns subset R_i of signals (which does not include shareholder i'signal θ_i) and does not know all the other signals, $-R_i$. Then, shareholder i reports his signal truthfully if and only if*

$$\begin{array}{ll} \textit{for } \rho_i \leq \rho_m: & \rho_m - \rho_i \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma - 1)\frac{\tau-\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|}, \\ \textit{for } \rho_i > \rho_m: & \rho_i - \rho_m \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma - 1)\frac{\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|}. \end{array} \tag{70}$$

This result shows that the overconfidence of each shareholder in the importance of his signal increases his incentives to report truthfully: the right-hand side of (70) increases in γ. Intuitively, if the shareholder perceives his signal to be more important than it actually is, he perceives misreporting to be costlier.

In addition, importantly, the communication decisions of the shareholders are complements: as in the basic model: the right-hand side of (70) increases as R_i expands. Overall, the key requirement for the complementarity effect is that communication of other shareholders to the manager moves the manager's and the shareholders's beliefs about the state closer to each other, which is consistent with heterogeneous interpretations of signals and does not require complete convergence of beliefs under full information. This property holds in a large class of models of different beliefs, although not in all of them.

A.6 Substitutability of shareholders' signals

In this section, we explore whether shareholders' signals in our setting are substitutes or complements under the definition of Borgers et al. (2013) and ask whether it matters for the externalities in communication.

A.6.1 Are signals complements or substitutes?

Under the definition of Borgers et al. (2013), signals are substitutes (complements) if the added effect of an additional signal on the agent's utility decreases (increases) with the number of signal, assuming the agent takes the optimal action given this information. Consider any investor j in our model with a prior belief that $\varphi \sim Beta(\rho_j, \tau - \rho_j)$. Suppose the investor knows the set R of signals. Using (6), the agent's optimal action is

$$a_j(\theta_R) = \sum_{i \in R} \theta_i + \frac{\rho_j + \sum_{i \in R} \theta_i}{\tau + |R|} (K - |R|) .$$

To calculate the agent's utility from knowing $|R|$ signals, $V_j(|R|)$, we rely on Lemma OA.2 in Section A.1 of the Online Appendix. Adapting the derivations behind Lemma 2, we get

$$V_j(|R|) = u_0 - \frac{\rho_j(\tau - \rho_j)}{\tau(\tau+1)} \left(K - |R| + \frac{(K - |R|)^2}{\tau + |R|} \right) = u_0 - \frac{\rho_j(\tau - \rho_j)}{\tau(\tau+1)} \frac{K - |R|}{\tau + |R|} (\tau + K) . \quad (71)$$

Consider $\mathcal{G}(r) \equiv \frac{K-r}{\tau+r}$. Since $\mathcal{G}'(r) < 0$ and $\mathcal{G}''(r) > 0$, we have $V_j(|R|)$ increasing and concave in $|R|$. Hence, $V_j(|R|) - V_j(|R| - 1)$ decreases in $|R|$, i.e., the signals are substitutes under the definition of Borgers et al. (2013).

A.6.2 What if signals are unconditionally independent?

To explore the role of the property that signals are substitutes, we change the assumption that φ is unknown and agents form beliefs about it. Instead, we assume that φ is a commonly known parameter. In particular, $Z = \sum_{i=1}^{K} \theta_i$,where investor i observes θ_i, and $\theta_1, ..., \theta_K$ are independent binary signals equal to 1 with probability φ and 0 otherwise, where φ is known.

In this case, if investor j know the set R of signals, his optimal action is

$$a_j^{ind}(\theta_R) = \mathbb{E}(Z \mid \theta_R) = \sum_{i \in R} \theta_i + (K - |R|) \varphi.$$

Hence, the agent's utility from knowing $|R|$ signals, $V_j^{ind}(|R|)$, is now given by

$$\begin{aligned} V_j^{ind}(|R|) &= u_0 - \mathbb{E}\left[\left(a_j^{ind}(\theta_R) - Z\right)^2 | \, \theta_R\right] = u_0 - \mathbb{E}\left[\left(\sum_{i\in R}\theta_i + (K-|R|)\,\varphi - \sum_{i=1}^{K}\theta_i\right)^2 | \, \theta_R\right] \\ &= u_0 - \mathbb{E}\left[\left(\sum_{i\in -R}(\theta_i - \varphi)\right)^2 | \, \theta_R\right] = u_0 - (K-|R|)\,Var\,(\theta_i)\,, \end{aligned}$$

where the last equality used the fact that θ_i are unconditionally independent and have mean φ. Hence, the utility from an additional signal is $Var\,(\theta_i)$ and does not depend on the number of signals the agent has, i.e., signals are neither complements nor substitutes under the definition of Borgers et al. (2013).

In this setting, we derive each shareholder's IC constraint for truthful communication. Suppose the manager knows signals in set R_i. If shareholder i reports his signal truthfully, the manager' action is

$$a_m^{ind}(\theta_{R_i}, \theta_i) \equiv b + \theta_i + \sum_{j\in R_i}\theta_j \ + \varphi\,(K - |R_i| - 1)\,. \tag{72}$$

If shareholder i misreports, the manager's action is

$$a_m^{ind}(\theta_{R_i}, 1-\theta_i) \equiv b + (1-\theta_i) + \sum_{j\in R_i}\theta_j \ + \varphi\,(K - |R_i| - 1)\,. \tag{73}$$

The shareholder compares his expected payoff from actions $a_m^{ind}(\theta_{R_i}, \theta_i)$ and $a_m^{ind}(\theta_{R_i}, 1-\theta_i)$ and reports his signal truthfully if and only if:

$$\sum_{\theta_{-i}\in\{0,1\}^{K-1}} \left[(a_m^{ind}(\theta_{R_i}, \theta_i) - Z)^2 - (a_m^{ind}(\theta_{R_i}, 1-\theta_i) - Z)^2\right] P(\theta_{-i}) \leq 0, \tag{74}$$

where $P(\theta_{-i})$ is his belief about θ_{-i}. Note that

$$(a_m^{ind}(\theta_{R_i}, \theta_i) - Z) - (a_m^{ind}(\theta_{R_i}, 1-\theta_i) - Z) = 2\theta_i - 1,$$

and

$$\begin{aligned} &(a_m^{ind}(\theta_{R_i}, \theta_i) - Z) + (a_m^{ind}(\theta_{R_i}, 1-\theta_i) - Z) = \\ &= 2b + 1 + 2\textstyle\sum_{j\in R_i}\theta_j \ + 2\varphi\,(K - |R_i| - 1) - 2\theta_i - 2\sum_{j\neq i}\theta_j \ . \end{aligned}$$

Hence, the shareholder reports truthfully if and only if

$$\begin{aligned}
(2\theta_i - 1) \sum_{\theta_{-i} \in \{0,1\}^{K-1}} \left(2b + 1 + 2\sum_{j \in R_i} \theta_j \; + 2\varphi\left(K - |R_i| - 1\right) - 2\theta_i - 2\sum_{j \neq i} \theta_j \right) P(\theta_{-i}) &\leq 0 \Leftrightarrow \\
(2\theta_i - 1)\left(2b + 1 + 2\left|R_i\right|\varphi + 2\varphi\left(K - |R_i| - 1\right) \; - 2\theta_i - 2\left(K-1\right)\varphi\right) &\leq 0 \Leftrightarrow \\
(2\theta_i - 1)\left(2b + 1 - 2\theta_i \;\right) &\leq 0.
\end{aligned}$$

If $\theta_i = 0$, the above inequality becomes $2b + 1 \geq 0$, which is always satisfied, and if $\theta_i = 1$, it becomes $b \leq \frac{1}{2}$. Hence, the IC condition no longer depends on $|R_i|$.

A.7 Single vs. multiple dimensions of expertise

Our basic model features multiple dimensions of expertise: while signals θ_i all provide noisy information about φ, each of them also provides incremental information about the state Z (since θ_i are independent conditional on φ). In this section, we explore the role of this assumption by comparing our results to those in a setting with one dimension of expertise.

Specifically, suppose that agent i's belief is that state Z is a draw from the Beta distribution with parameters $(\rho_i, \tau - \rho_i)$. Each shareholder i obtains a signal $\theta_i \in \{0, 1\}$ about Z, where $\theta_i = 1$ with probability Z and $\theta_i = 0$ with probability $1 - Z$.

Suppose that after the communication stage, the manager knows subset R of signals. Then his optimal action is

$$a_m(\theta_R) = b + \mathbb{E}_m\left(Z \,|\, \theta_R\right) = b + \frac{\rho_m + \sum_{i \in R} \theta_i}{\tau + |R|}. \tag{75}$$

Suppose the manager believes the shareholder's message and uses it to update his belief about the state according to (75). If shareholder i reveals his signal truthfully, the manager's action is

$$a_m\left(\theta_{R_i}, \theta_i\right) \equiv b + \frac{\rho_m + \theta_i + \sum_{j \in R_i} \theta_j}{\tau + |R_i| + 1}. \tag{76}$$

If shareholder i misreports and claims that his signal is $1 - \theta_i$, the manager's action is

$$a_m\left(\theta_{R_i}, 1 - \theta_i\right) \equiv b + \frac{\rho_m + (1 - \theta_i) + \sum_{j \in R_i} \theta_j}{\tau + |R_i| + 1}. \tag{77}$$

Note that

$$\begin{aligned}
a_m\left(\theta_{R_i}, \theta_i\right) - a_m\left(\theta_{R_i}, 1 - \theta_i\right) &= \frac{2\theta_i - 1}{\tau + |R_i| + 1}, \\
a_m\left(\theta_{R_i}, \theta_i\right) + a_m\left(\theta_{R_i}, 1 - \theta_i\right) - 2Z &= 2b + \frac{2\rho_m + 1 + 2\left(\sum_{j \in R_i} \theta_j\right)}{\tau + |R_i| + 1} - 2Z.
\end{aligned}$$

Hence, the IC constraint for shareholder i to communicate truthfully is:

$$-\int_0^1 \sum_{\theta_{R_i}} \frac{2\theta_i - 1}{\tau + |R_i| + 1} \left(2b + \frac{2\rho_m + 1 + 2\left(\sum_{j\in R_i} \theta_j\right)}{\tau + |R_i| + 1} - 2Z \right) f_i(Z, \theta_{R_i}|\theta_i) dZ \geq 0, \tag{78}$$

where $f_i(Z, \theta_{R_i}|\theta_i)$ is the shareholder's belief given his signal θ_i and prior beliefs. Simplifying:

$$\begin{aligned} -(2\theta_i - 1)\left(2b + \int_0^1 \sum_{\theta_{R_i}} \left(\frac{2\rho_m + 1 + 2\left(\sum_{j\in R_i} \theta_j\right)}{\tau + |R_i| + 1} - 2Z \right) f_i(Z, \theta_{R_i}|\theta_i) dZ \right) &\geq 0 \Leftrightarrow \\ -(2\theta_i - 1)\left(2b + \frac{2\rho_m + 1 + 2|R_i|\mathbb{E}_i[\theta_j|\theta_i]}{\tau + |R_i| + 1} - 2\mathbb{E}_i[Z|\theta_i] \right) &\geq 0. \end{aligned}$$

Note that $\mathbb{E}_i[Z|\theta_i] = \frac{\rho_i + \theta_i}{\tau+1}$ and $\mathbb{E}_i[\theta_j|\theta_i] = \mathbb{E}_i[Z|\theta_i] = \frac{\rho_i+\theta_i}{\tau+1}$. Therefore, the above expression simplifies to:

$$\begin{aligned} -(2\theta_i - 1)\left(2b + \frac{2\rho_m + 1}{\tau + |R_i| + 1} + \frac{2\mathbb{E}_i[Z|\theta_i]}{\tau + |R_i| + 1}(|R_i| - \tau - |R_i| - 1) \right) &\geq 0 \Leftrightarrow \\ -(2\theta_i - 1)\left(2b + \frac{2\rho_m + 1}{\tau + |R_i| + 1} - \frac{2(\rho_i + \theta_i)}{\tau + |R_i| + 1} \right) &\geq 0 \Leftrightarrow \\ -(2\theta_i - 1)\left(2b + \frac{2(\rho_m - \rho_i)}{\tau + |R_i| + 1} + \frac{1 - 2\theta_i}{\tau + |R_i| + 1} \right) &\geq 0. \end{aligned}$$

There are two cases to consider, $\theta_i = 1$ and $\theta_i = 0$. When $\theta_i = 1$, the IC constraint is:

$$2b + \frac{2(\rho_m - \rho_i)}{\tau + |R_i| + 1} - \frac{1}{\tau + |R_i| + 1} \leq 0.$$

When $\theta_i = 0$, the IC constraint is:

$$2b + \frac{2(\rho_m - \rho_i)}{\tau + |R_i| + 1} + \frac{1}{\tau + |R_i| + 1} \geq 0$$

Together, we get

$$2\left| b + \frac{\rho_m - \rho_i}{\tau + |R_i| + 1} \right| \leq \frac{1}{\tau + |R_i| + 1}. \tag{79}$$

As in the basic model, the left-hand side of (79) captures the incongruence between the manager and the shareholder, whereas the right-hand side of (79) measures the manager's reaction to the shareholder's advice (i.e., by how much the manager's action changes if the shareholder misreports his signal θ_i, as can be seen from (76) and (77)). Hence, both the complementarity and the substitution effects again arise in this setting. The complementarity effect is represented by the term $\frac{\rho_m - \rho_i}{\tau+|R|+1}$: as the manager learns more signals from other shareholders, his posterior beliefs become closer to shareholder i's posterior beliefs. This increases the congruence between them (as captured by a decrease in the left-hand side of (79)) and increases the shareholder's incentives to communicate truthfully. The substitution effect is represented by the term $\frac{1}{\tau+|R_i|+1}$: as the manager learns more signals from other shareholders, he reacts less to the shareholder's advice, making misreporting more appealing. Finally, the property that a stronger misalignment of preferences limits the complementarity effect also continues to hold: as b increases, the manager's learning has a relatively smaller effect on the congruence between the manager and shareholder (as captured by the left-hand-side of (79) being increasingly limited from zero).

Note also that if $b = 0$, the effect of $|R_i|$ cancels out, i.e., the complementarity effect is fully offset by the substitution effect. While the complete offsetting of the two effects is a specific property of the Beta distribution, the more general intuition is that the substitution effect is relatively weaker in the presence of multiple dimensions of expertise (as in the basic model) than if there is a single dimension of expertise (as in the model in this section). To see why this is the case, let us compare the IC condition (79) in this setting to the corresponding IC condition in the basic model:

$$2\left|b + \frac{K - |R_i| - 1}{\tau + |R_i| + 1}(\rho_m - \rho_i)\right| \leq 1 + \frac{K - |R_i| - 1}{\tau + |R_i| + 1}. \tag{80}$$

Relative to (79), the additional additive term 1 on the right-hand side of (80) weakens the substitution effect and arises due to the multi-dimensionality of shareholders' expertise: even if the manager strongly updates his beliefs about the common state φ (as captured by a decreasing function $\frac{1}{\tau+|R_i|+1}$), he still strongly reacts to the shareholder's message if it provides information about a new dimension of uncertainty (as captured by the term 1). This encourages truth-telling and weakens the substitution effect.

A.8 Communication among shareholders

Suppose that the firm is owned by S_o optimistic and S_p pessimistic shareholders. Consider the following change in the communication stage of the game. Instead of each shareholder independently sending a binary message to the manager, suppose that all shareholders of the same type (i.e., with the same prior beliefs) share their signals among themselves, and one representative of the group then communicates with the manager via cheap talk. Since shareholders' interests within a group are fully aligned, they share their information truthfully with each other. Given this change in the setup, we effectively have a two-sender model in which one sender (representing the optimists) has signal $\Theta_o \equiv \sum_{i=1}^{S_o} \theta_i$, taking discrete values from zero to S_o, and the other sender (representing the pessimists) has signal $\Theta_p \equiv \sum_{i=S_o+1}^{S_o+S_p} \theta_i$, taking discrete values from zero to S_p. Signals θ_i for $i \in \{S_o + S_p + 1, ..., K\}$ are unknown to everyone.

Let μ_o and μ_p denote the messages of the sender representing optimistic and pessimistic shareholders, respectively. In what follows, we will derive the necessary and sufficient conditions under which there exists a fully informative equilibrium, i.e., an equilibrium in which the representatives of each group communicate all information of the group truthfully: the sender representing optimistic (pessimistic) shareholders sends message $\mu_o = \Theta_o$ ($\mu_p = \Theta_p$). The next result shows that these conditions are the same as the necessary and sufficient conditions for the existence of a fully informative equilibrium (i.e., $|R| = S_o + S_p$) without communication among shareholders.

Proposition OA.5. *The necessary and sufficient conditions for the existence of a fully informative equilibrium in the model with communication through a representative are the same as the necessary and sufficient conditions for the existence of a fully informative equilibrium in the basic model.*

The logic of this result is based on two steps. The first step is that the costs and benefits of a sender deviating from truthful communication *by one unit* (e.g., from Θ_o to $\Theta_o - 1$ or to $\Theta_o + 1$) are the same in the basic model and in the model with communication through a representative. This is because the action of the manager and the payoffs of all investors and the sender are the same in the basic model and in this extension, both if the sender tells the truth and if he deviates in his message by one unit. Thus, the IC constraints that make such deviations suboptimal are also the same. The second step concerns global deviations by the sender in the model with communication through a representative. In the model in which all shareholders communicate to the manager directly, each shareholder only has access to "small" deviations because each shareholder's signal is binary: for example, if a shareholder gets a signal of 0, he can misreport and say that his signal is 1, but he cannot misreport and say that his signal is 2. In contrast, if the representative communicates with the manager on behalf of all shareholders of the group, he can misreport by more than one unit. Nevertheless, we show that these additional deviations have no additional bite: if a deviation from truthful reporting by more than one unit is profitable to a sender, then a deviation in the same direction by one unit must also be profitable for a sender. Given these two steps, a fully informative equilibrium exists in the model with communication through a representative if and only if it exists in the basic model.[29]

A.9 Sequential communication

Suppose that the firm is owned by S shareholders. Consider the following change in the communication stage of the game. Instead of communicating simultaneously to the manager, there is sequential public communication: shareholders communicate in a known sequence $O = \{O_j, j \in S\}$ such that shareholder $j \in S$ is O_j-th to send a public message, which is observed by all the other shareholders and the manager. The next result characterizes the IC condition for truthful communication in this variation of the model.

[29] Characterizing the full set of equilibria in the model with communication through a representative is more challenging than in the basic model. The reason is that in the basic model, we use the property that if the prior distribution of φ is Beta, then the posterior distribution of φ conditional on learning a number of binary signals is also Beta (with parameters that depend on the realizations of the learned signals). In contrast, in a partially informative equilibrium of the extended game, the manager will only learn that the sum of all signals of shareholders of the same type lies in some set. As a consequence, the posterior distribution of φ is no longer the Beta distribution, which makes the analysis less tractable.

Proposition OA.6. *Consider the sequential game described above. Suppose shareholder i expects subset $R_i \subseteq S \backslash \{i\}$ of other shareholders to communicate truthfully and other shareholders to send uninformative messages. Then, for any sequence O, shareholder i has incentives to send a truthful message if and only if (10) is satisfied.*

Hence, the IC condition for truthful communication for each shareholder is exactly the same as in the basic model. Intuitively, what matters for the shareholder's incentives is the combined set of signals that the manager learns before taking his action, as this combined set of signals determines both the manager's reaction to the shareholder's advice and the congruence between the manager and the shareholder at the decision-making stage.

It follows that for any sequence of communication, the equilibrium at the communication stage is the same as in the model with simultaneous communication. Thus, if the expectation at the trading stage is that this communication equilibrium will be played, then the solution of the trading stage will also be unchanged, leading to the same equilibrium as in the basic model. Of course, the sequential game may also have other equilibria, which we do not analyze.

A.10 Incentives to become a passive investor

According to Proposition 5, the existence of passive investors improves communication and the share price. Given this, is it optimal for some investors to commit to being passive at the beginning of the game? In our current model, this is not the case because, although by committing to become a passive shareholder, an investor can improve communication and thereby increase the value of the firm, this value increase gets reflected in the higher stock price that the investor pays when buying shares.

To see this formally, consider the conditions of Proposition 5 (in particular, $\lambda < \hat{\lambda}$). First, notice that from the perspective of an optimist, it is never optimal to commit to being passive: optimists become shareholders regardless of whether they are active or passive, so committing to becoming passive has no effect on the quality of communication and only constrains the optimist's investment decisions. We next prove that from the perspective of a pessimist, it is not optimal to commit to being passive either. This is because a pessimist who commits to holding $\frac{1}{N}$ shares gets expected utility of

$$\frac{1}{N}(\mathbb{E}_p[U|\hat{r}_2(L)] - p(L)) - \frac{\lambda}{2N^2}, \tag{81}$$

where $\hat{r}_2(L)$ is (as defined in the proof of Proposition 5) the equilibrium number of shareholders who communicate truthfully if there are L passive investors, and $p(L)$ is the equilibrium stock price given by (50): $p(L) = \mathbb{E}_o[U|\hat{r}_2(L)] - \frac{\lambda}{N_o}$. Using the expression for $p(L)$, we have:

$$\mathbb{E}_p[U|\hat{r}_2(L)] \geq p(L) \Leftrightarrow \mathbb{E}_o[U|\hat{r}_2(L)] - \mathbb{E}_p[U|\hat{r}_2(L)] \leq \frac{\lambda}{N_o}, \tag{82}$$

which is equivalent to (24) when $|R| = \hat{r}_2(L)$. According to the proof of Proposition 4, if $\lambda < \hat{\lambda}$, then (24) is violated for any R such that $|R| \leq \hat{r}$. Since $\hat{r}$ is the number of signals communicated to the manager if all investors become shareholders, we have $\hat{r} \geq \hat{r}_2(L)$, and hence (24) is violated if $|R| = \hat{r}_2(L)$. Thus, (82) is violated, which implies that the pessimist's utility (81) from committing to being passive is strictly negative. Deviation to being active is therefore profitable for the pessimist because under an active investment strategy, he does not acquire any shares and gets utility of zero.

For passive ownership to arise endogenously, we need to have a mechanism through which the investor could capture a sufficient amount of value he creates by improving communication. One possible mechanism is to assume that investors have some endowment of shares. To see how passive ownership can arise endogenously in such a setting, consider the following variation of the baseline model. Suppose each shareholder has an endowment of $\frac{\alpha}{N}$ shares at the start of the game. The remaining $1-\alpha$ of the shares are held by a third party, which sells them. We assume that shareholders' preferences are the same as in the baseline model. In particular, shareholder i's utility from holding α_i shares after the trading stage is given by (3):

$$\alpha_i \left(\mathbb{E}_i[U(a, Z)] - p\right) - \frac{\lambda}{2}\alpha_i^2.$$

This implies, in particular, that in this extension, λ captures the holding costs due to limited diversification and risk aversion (which applies to the entire ownership stake α_i after trading), rather than transaction costs (which would only apply to the incremental amount traded).

Since shareholders' preferences are the same as in the baseline model, the equilibrium ownership stakes and the equilibrium at the communication stage remain the same. The only difference is in the expected payoffs of the shareholders, since they now have an initial endowment. Suppose that at the beginning of the game, each shareholder can announce that he will be a passive investor and thereby commit that he will hold $\frac{1}{N}$ shares regardless of his valuation. For the same reason as above, the optimists will never find it optimal to commit to being passive (because this does not improve communication and only constrains their trading choices), and hence such commitment can only be optimal for pessimists.[30]

In this extension, if α is sufficiently high, passive ownership will arise endogenously, and the magnitude of α will determine how many investors become passive funds. We show how this happens using an example.

Example. Consider the following parameters: $N = 6$, $N_o = 3$, $\rho = 1$ (i.e., the manager's prior is that φ is uniform on $[0, 1]$), $\Delta = 0.5$, $b = 0.1$, $K = 10$, $\lambda = 0.2$, $u_0 = 5$. The detailed derivations are provided in Section A.11.9 below, and we only summarize the results here.

First, it is easy to check that the constraint from Proposition 5 is satisfied. If there are no passive funds ($L = 0$), then after the trading stage, $N_o = 3$ optimistic investors hold shares of the firm ($\frac{1}{3}$ of the share each) and communicate truthfully, while pessimistic investors do not hold shares (i.e., they sell their initial endowments) and do not communicate. The equilibrium

[30] Hence, in this extension, we no longer impose the assumption that optimists and pessimists are equally represented among passive and active investors (which we make in Section 4.2 of the paper), and we recompute the prices accordingly. An alternative would be to assume that investors commit to becoming passive before they learn whether they are optimists or pessimists, which would produce similar results with somewhat more complicated derivations.

stock price is then 2.47, and the expected valuations of the pessimists and optimists are, respectively, $\mathbb{E}_p[U|N_o] = 2.26$ and $\mathbb{E}_o[U|N_o] = 2.54$.

Second, suppose that one pessimistic investor announces that he will be a passive shareholder, thereby committing to owning $\frac{1}{6}$ of the share, irrespectively of his valuation of the shares. In equilibrium, the remaining $\frac{5}{6}$ of the share will be held by the three optimistic investors in equal proportions ($\frac{5}{18}$ each), and all four shareholders will truthfully communicate to the manager. The equilibrium stock price increases to 3.28, and investors' valuations become $\mathbb{E}_p[U|N_o+1] = 3.14$ and $\mathbb{E}_o[U|N_o+1] = 3.34$. For the pessimistic investor to find it optimal to commit to being a passive shareholder, α needs to be sufficiently high, so that the gain from the value increase of his endowment outweighs the loss from paying above his valuation for the shares. We find that for the parameters above, this is satisfied if and only if $\alpha \geq 0.1986$. If, in addition, $\alpha < 0.2157$, then exactly one investor finds it optimal to commit to being a passive shareholder, i.e., it is suboptimal for another investor to do so. If, in contrast, $\alpha > 0.2157$, then there will be more than one passive investor in equilibrium.

To summarize, although in the baseline model it is suboptimal for an investor to commit to being passive due to the fact that the price appreciation from more effective communication accrues to the initial owners, in a natural extension in which investors have some endowment of shares, committing to being passive can be optimal, and the number of passive investors can be endogenized.

A.11 Proofs for the Online Appendix

A.11.1 Proof of Lemma OA.1

Since θ_i is a binary signal equal to 1 with probability φ and 0 with probability $1-\varphi$, the manager's optimal action can be written as:

$$a_m(\theta_R) = b + \sum_{i\in R} c_i\theta_i + \mathbb{E}_m\left[\varphi|\theta_i, i\in R\right]\sum_{j\in -R} c_j \ .$$

Let $\mathbf{1}_R \equiv \sum_{i\in R}\theta_i$ be the number of signals in R equal to 1. The conditional probability that $\mathbf{1}_R$ signals out of $|R|$ are equal to one given φ is $P(\mathbf{1}_R|\varphi) = \binom{|R|}{\mathbf{1}_R}\varphi^{\mathbf{1}_R}(1-\varphi)^{|R|-\mathbf{1}_R}$. Since the prior distribution is Beta and the likelihood function is Binomial, the posterior distribution is also Beta but with different parameters (this is a known property of the Beta distribution). Formally, let $P_i(\mathbf{1}_R)$ be agent i's assessed probability that $\mathbf{1}_R$ signals out of $|R|$ are equal to 1 (over all possible values of φ). Using Bayes rule, agent i's posterior belief of φ, $P_i(\varphi|\mathbf{1}_R)$, is

$$\begin{aligned}
P_i(\varphi|\mathbf{1}_R) &= \frac{f_i(\varphi)P(\mathbf{1}_R|\varphi)}{P_i(\mathbf{1}_R)} = \frac{\varphi^{\rho_i-1}(1-\varphi)^{\tau-\rho_i-1}}{Beta(\rho_i,\tau-\rho_i)}\frac{1}{P_i(\mathbf{1}_R)}\binom{|R|}{\mathbf{1}_R}\varphi^{\mathbf{1}_R}(1-\varphi)^{|R|-\mathbf{1}_R} \\
&= \frac{1}{Beta(\rho_i,\tau-\rho_i)P_i(\mathbf{1}_R)}\binom{|R|}{\mathbf{1}_R}\times\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1},
\end{aligned}$$

which is some constant that does not depend on φ times $\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1}$. Since the posterior beliefs must integrate to one over possible values of φ, this automatically implies that the posterior belief also follows a Beta distribution with parameters $(\rho_i + \mathbf{1}_R, \tau - \rho_i + |R| - \mathbf{1}_R)$ and density

$$P_i(\varphi|\mathbf{1}_R) = \frac{1}{Beta(\rho_i + \mathbf{1}_R, \tau - \rho_i + |R| - \mathbf{1}_R)} \varphi^{\rho_i + \mathbf{1}_R - 1}(1-\varphi)^{\tau - \rho_i + |R| - \mathbf{1}_R - 1}.$$

It is known that the mean of a Beta distribution with parameters (α, β) is $\frac{\alpha}{\alpha+\beta}$. Therefore, using these expressions and the above posterior distribution, agent i's expected value of φ is $\mathbb{E}_i(\varphi|\mathbf{1}_R) = \frac{\rho_i + \mathbf{1}_R}{\tau + |R|}$, which proves the lemma.

A.11.2 Proof of Lemma OA.2

Let $\mathbf{1}_R = \sum_{i \in R} \theta_i$ denote the number of signals 1 in R. Using Lemma OA.1, we obtain investor i's ex-ante payoff, $\mathbb{E}_i(a_m(\theta_R) - Z)^2$, as follows:

$$\mathbb{E}_i [U_i|R] = u_0 - b^2 - U_1 - U_2, \tag{83}$$

where

$$\begin{aligned} U_1 &\equiv 2b\mathbb{E}_i \left[\left(\frac{\rho_m + \mathbf{1}_R}{\tau + |R|} \sum_{j \in -R} c_j - \sum_{j \in -R} c_j \theta_j \right) |R \right], \\ U_2 &\equiv \mathbb{E}_i \left[\left(\frac{\rho_m + \mathbf{1}_R}{\tau + |R|} \sum_{j \in -R} c_j - \sum_{j \in -R} c_j \theta_j \right)^2 |R \right]. \end{aligned}$$

Using independence of θ_j conditional on φ, and Auxiliary Lemma A.1, U_1 simplifies to

$$U_1 = 2b \frac{\rho_m - \rho_i}{\tau + |R|} \left(\sum_{j \in -R} c_j \right) = \mathsf{A}_{im}(R). \tag{84}$$

To simplify U_2, we use the law of iterated expectations:

$$\begin{aligned} U_2 &= \mathbb{E}_i \left[\left(\frac{(\rho_m + \mathbf{1}_R) \sum_{j \in -R} c_j}{\tau + |R|} \right)^2 - 2 \frac{(\rho_m + \mathbf{1}_R)(\rho_i + 1_R) \left(\sum_{j \in -R} c_j \right)^2}{(\tau + |R|)^2} |R \right] \\ &+ \mathbb{E}_i \left[\mathbb{E}_i \left[\left(\sum_{j \in -R} c_j \theta_j \right)^2 |\theta_R, R \right] |R \right], \end{aligned} \tag{85}$$

where we used $\mathbb{E}_i\left[\sum_{j\in -R} c_j\theta_j|\theta_R,R\right] = \left(\sum_{j\in -R} c_j\right)\mathbb{E}_i\left[\varphi|\theta_R,R\right] = \left(\sum_{j\in -R} c_j\right)\frac{\rho_i+1_R}{\tau+|R|}$. Consider the last term under the expectation sign:

$$\begin{aligned}
\mathbb{E}_i\left[\left(\sum_{j\in -R} c_j\theta_j\right)^2|\theta_R,R\right] &= \mathbb{E}_i\left[\sum_{j\in -R} c_j^2 Var_i\left[\theta_j|\varphi,R\right] + \varphi^2\left(\sum_{j\in -R} c_j\right)^2|\theta_R,R\right] \\
&= \mathbb{E}_i\left[\sum_{j\in -R} c_j^2\varphi\left(1-\varphi\right) + \varphi^2\left(\sum_{j\in -R} c_j\right)^2|\theta_R,R\right] \\
&= \frac{\rho_i+1_R}{\tau+|R|}\left(\sum_{j\in -R} c_j^2 + \left(\left(\sum_{j\in -R} c_j\right)^2 - \sum_{j\in -R} c_j^2\right)\frac{\rho_i+1_R+1}{\tau+|R|+1}\right),
\end{aligned}$$

where the second equality is due to $Var_i\left[\theta_j|\varphi,R\right] = \varphi\left(1-\varphi\right)$ and the last equality is due to the fact that the agent i's posterior distribution of φ conditional on θ_R is Beta with parameters ρ_i+1_R and $\tau+|R|-\rho_i-1_R$, whose first and second moments are, respectively, $\frac{\rho_i+1_R}{\tau+|R|}$ and $\frac{(\rho_i+1_R)(\rho_i+1_R+1)}{(\tau+|R|)(\tau+|R|+1)}$. Plugging this expression into (85) and simplifying using Auxiliary Lemma A.1,

$$\begin{aligned}
U_2 - \mathsf{C}_{im}(R) &= \mathbb{E}_i\left[\frac{\left(\sum_{j\in -R} c_j^2\right)\left(\rho_i+1_R\right)}{\tau+|R|} - \left(\frac{\left(\sum_{j\in -R} c_j\right)\left(\rho_i+1_R\right)}{\tau+|R|}\right)^2|R\right] \\
&\quad + \left(\left(\sum_{j\in -R} c_j\right)^2 - \sum_{j\in -R} c_j^2\right)\mathbb{E}_i\left[\frac{\left(\rho_i+1_R+1\right)\left(\rho_i+1_R\right)}{\left(\tau+|R|+1\right)\left(\tau+|R|\right)}|R\right] \\
&= \left(\frac{\left(\sum_{j\in -R} c_j\right)^2}{\tau+|R|} + \sum_{j\in -R} c_j^2\right)\mathbb{E}_i\left[\frac{\left(\rho_i+1_R\right)\left(\tau+|R|-\rho_i-1_R\right)}{\left(\tau+|R|\right)\left(\tau+|R|+1\right)}\right] \\
&= \left(\frac{\left(\sum_{j\in -R} c_j\right)^2}{\tau+|R|} + \sum_{j\in -R} c_j^2\right)\frac{\rho_i\left(\tau-\rho_i\right)}{\tau\left(\tau+1\right)} = \mathsf{B}_i(R).
\end{aligned}$$

Combining with (83) and (84) completes the proof.

A.11.3 Proof of Proposition OA.1

Suppose the manager believes the shareholder's message and uses it to update his belief about the state. If shareholder i reveals his signal truthfully, the manager's action is

$$a_m\left(\theta_R,\theta_i\right) \equiv b + c_i\theta_i + \sum_{j\in R} c_j\theta_j + \frac{\rho_m+\theta_i+\sum_{j\in R}\theta_j}{\tau+1+|R|}\sum_{j\in -R\setminus\{i\}} c_j. \tag{86}$$

In contrast, if shareholder i misreports and claims that his signal is $1-\theta_i$, the manager's action is

$$a_m\left(\theta_R, 1-\theta_i\right) \equiv b + c_i\left(1-\theta_i\right) + \sum_{j\in R} c_j\theta_j \; + \frac{\rho_m + (1-\theta_i) + \sum_{j\in R}\theta_j}{\tau + 1 + |R|} \sum_{j\in -R\setminus\{i\}} c_j. \tag{87}$$

Because shareholder i does not know the realization of all other agents' ($N-1$ shareholders' and the manager's) signals, he compares his expected payoff from actions $a_m\left(\theta_R, \theta_i\right)$ and $a_m\left(\theta_R, 1-\theta_i\right)$ given his signal θ_i and his own prior belief about the distribution of those other N signals, and reports his signal truthfully if and only if:

$$\sum_{\theta_{-i}\in\{0,1\}^N} \left[\left(a_m\left(\theta_R, \theta_i\right) - Z\right)^2 - \left(a_m\left(\theta_R, 1-\theta_i\right) - Z\right)^2\right] P_i(\theta_{-i}|\theta_i) \leq 0, \tag{88}$$

where θ_{-i} is the set of all signals except θ_i and $P_i\left(\theta_{-i}|\theta_i\right)$ is shareholder i's belief given his signal θ_i and his own prior.

Plugging (86) and (87) into (88) gives

$$\begin{aligned} 0 \geq \sum_{\theta_{-i}\in\{0,1\}^N} & \left[c_i(2\theta_i - 1) + \Big(\sum_{j\in -R_i\setminus\{i\}} c_j\Big)\cdot\frac{2\theta_i - 1}{\tau + |R_i| + 1}\right] \\ & \times \left[2b + c_i(1-2\theta_i) - 2\sum_{j\in -R_i\setminus\{i\}} c_j\theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1}\sum_{j\in -R_i\setminus\{i\}} c_j\right] P_i(\theta_{-i}|\theta_i). \end{aligned}$$

Note that the first multiple in each term equals $(2\theta_i - 1)[c_i + \frac{\sum_{j\in -R_i\setminus\{i\}} c_j}{\tau+|R_i|+1}]$, where $c_i + \frac{\sum_{j\in -R_i\setminus\{i\}} c_j}{\tau+|R_i|+1}$ is positive and is constant across all terms in the sum. Thus, the above inequality is equivalent to

$$0 \geq (2\theta_i - 1)\sum_{\theta_{-i}} P_i(\theta_{-i}|\theta_i)\left(2b + c_i(1-2\theta_i) - 2\sum_{j\in -R_i\setminus\{i\}} c_j\theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1}\sum_{j\in -R_i\setminus\{i\}} c_j\right).$$

Since $\sum_{\theta_{-R_i\setminus\{i\}}}\left(\sum_{j\in -R_i\setminus\{i\}} c_j\theta_j\right) P_i(\theta_{-R_i\setminus\{i\}}|\theta_i, \theta_{R_i}) = \frac{\rho_i + \mathbf{1}_{R_i} + \theta_i}{\tau+|R_i|+1}\sum_{j\in -R_i\setminus\{i\}} c_j$, we can further simplify it to

$$(2\theta_i - 1)\left[2b + c_i(1-2\theta_i) + \frac{2(\rho_m - \rho_i) + 1 - 2\theta_i}{\tau + |R_i| + 1}\sum_{j\in -R_i\setminus\{i\}} c_j\right] \leq 0.$$

We consider two separate cases. If $\theta_i = 0$, the above inequality becomes:

$$2b + c_i + \frac{2(\rho_m - \rho_i) + 1}{\tau + |R_i| + 1} \sum_{j \in -R_i \setminus \{i\}} c_j \geq 0,$$

and if $\theta_i = 1$, it becomes

$$2b - c_i + \frac{2(\rho_m - \rho_i) - 1}{\tau + |R_i| + 1} \sum_{j \in -R_i \setminus \{i\}} c_j \leq 0,$$

Together we get (58), which completes the proof.

A.11.4 Proof of Proposition OA.2

The proof largely repeats the proof of Proposition 1 in the paper. Using $a_m(\theta_{R_i}, \theta_i)$ and $a_m(\theta_{R_i}, 1 - \theta_i)$, the IC becomes

$$\begin{gathered}\sum_{\theta_{R_i}, \theta_{-R_i}} P_i(\theta_{R_i}, \theta_{-R_i} | \theta_i) \left[c_i(2\theta_i - 1) + \left(\sum_{j \in -R_i \setminus \{i\}} c_j \right) \tfrac{2\theta_i - 1}{\tau_m + |R_i| + 1} \right] \times \\ \left[2b + c_i(1 - 2\theta_i) - 2 \sum_{j \in -R_i \setminus \{i\}} c_j \theta_j + (\sum_{j \in -R_i \setminus \{i\}} c_j) \tfrac{2(\rho_m + \sum_{j \in R_i} \theta_j) + 1}{\tau_m + |R_i| + 1} \right] \geq 0.\end{gathered}$$

Note that $P_i(\theta_{R_i}, \theta_{-R_i} | \theta_i) = P_i(\theta_{-R_i} | \theta_{R_i}, \theta_i) P_i(\theta_{R_i} | \theta_i)$. Since $\left[c_i + \frac{\sum_{j \in -R_i \setminus \{i\}} c_j}{\tau_m + |R_i| + 1} \right] > 0$, this is equivalent to

$$-(2\theta_i - 1) \times \left[2b + c_i(1 - 2\theta_i) - 2 \frac{\rho_i + \theta_i}{\tau_i + 1} \sum_{j \in -R_i \setminus \{i\}} c_j + (\sum_{j \in -R_i \setminus \{i\}} c_j) \frac{2\rho_m + 2 |R_i| \frac{\rho_i + \theta_i}{\tau_i + 1} + 1}{\tau_m + |R_i| + 1} \right] \geq 0$$

or equivalently,

$$-(2\theta_i - 1) \times \left[2b + c_i(1 - 2\theta_i) + (\sum_{j \in -R_i \setminus \{i\}} c_j) \left[\frac{2\rho_m(\tau_i + 1) + \tau_i + 1 - 2(\rho_i + \theta_i)(\tau_m + 1)}{(\tau_i + 1)(\tau_m + |R_i| + 1)} \right] \right] \geq 0$$

Considering two cases ($\theta_i = 1$ and $\theta_i = 0$) and simplifying the expressions (similar to the proof of Proposition 1), we obtain (63). It is easy to see that (63) is equivalent to the condition (58), which was derived for the same setting but with $\tau_i = \tau$ for all i.

A.11.5 Proof of Proposition OA.3

Let $V_i(r)$ denote investor i's payoff before acquiring and learning his private signal, which is given by (12) with $|R| = r$. Let S be the firm's shareholder base. Suppose there is an equilibrium in which all shareholders in S acquire information, which, in turn, requires that they communicate it truthfully. Consider the shareholder's decision to acquire information in this equilibrium. If shareholder i acquires his signal, his expected utility is $\alpha_i V_i(|S|) - \kappa$. If the shareholder deviates and does not acquire his signal, his expected utility is $\alpha_i V_i(|S|-1)$: because information acquisition decisions are observed after the communication stage, the shareholder's deviation does not change other shareholder' incentives to communicate truthfully, but at the decision-making stage, the manager will make his decision knowing that the shareholder is uninformed. Hence, the incentive compatibility condition on information acquisition by all shareholders is that:

$$V_i(|S|) - V_i(|S|-1) \geq \frac{\kappa}{\alpha_i} \tag{89}$$

for all i. To analyze these conditions, consider the function $V_i(r) - V_i(r-1)$. Using (12) and denoting $\mathcal{G}(r) \equiv \frac{K-r}{2\rho+r}$, we get

$$V_i(r) = u_0 - b^2 - 2b(\rho-\rho_i)\mathcal{G}(r) - \frac{\rho^2-\Delta^2}{2\rho(2\rho+1)}\mathcal{G}(r)(2\rho+K) - [\Delta\mathcal{G}(r)]^2. \tag{90}$$

Note that $\mathcal{G}(r) > 0$, and that $\mathcal{G}(r)$ and hence $\mathcal{G}^2(r)$ decrease in r. In addition, $\mathcal{G}''(r) > 0$, and hence $(\mathcal{G}^2)''(r) > 0$ as well. It follows that the function $V_i(r)$ is increasing and concave in r, and hence $V_i(r) - V_i(r-1)$ is decreasing in r. If $b = 0$, then optimists and pessimists have the same valuations of the stock and hence hold the same number of shares, i.e., $S = \{1, ..., N\}$, $V_i(r) = V(r)$ for all i, and $\alpha_i = \frac{1}{N}$. Hence, (89) is equivalent to $\frac{V_i(N)-V_i(N-1)}{N} \geq \kappa$. Since $V_i(r) - V_i(r-1)$ is decreasing in r, $\frac{V_i(r)-V_i(r-1)}{r}$ is decreasing in r as well. Let $\hat{N}(\kappa)$ be the highest value of r for which $\frac{V_i(r)-V_i(r-1)}{r} \geq \kappa$, and note that $\hat{N}(\kappa)$ is weakly decreasing in κ. Then, the incentive compatibility condition on information acquisition is satisfied for all shareholders if and only if $N \leq \hat{N}(\kappa)$.

A.11.6 Proof of Proposition OA.4

Suppose that the manager expects shareholder i to report his signal truthfully, and consider shareholder i's decision whether to do so. If shareholder i reveals his signal truthfully, the manager's action is

$$a_m(\theta_{R_i}, \theta_i) \equiv \theta_i + \gamma\theta_m + \sum_{j \in R_i \setminus \{m\}} \theta_j + \frac{\rho_m + \theta_i + \sum_{j \in R_i} \theta_j}{\tau + 1 + |R_i|}(N - |R_i|). \tag{91}$$

In contrast, if shareholder i misreports, the manager's action is

$$a_m(\theta_{R_i}, 1-\theta_i) \equiv (1-\theta_i) + \gamma\theta_m + \sum_{j \in R_i \setminus \{m\}} \theta_j + \frac{\rho_m + (1-\theta_i) + \sum_{j \in R_i} \theta_j}{\tau + 1 + |R_i|}(N - |R_i|). \tag{92}$$

Truthful reporting is optimal if and only if

$$\sum_{\theta_{-i} \in \{0,1\}^N} \left[(a_m(\theta_{R_i}, \theta_i) - Z_i)^2 - (a_m(\theta_{R_i}, 1-\theta_i) - Z_i)^2\right] P_i(\theta_{-i}|\theta_i) \leq 0 \tag{93}$$

for each $\theta_i \in \{0,1\}$, where $Z_i = \gamma\theta_i + \sum_{j\neq i}\theta_j$. Simplifying, (93) reduces to

$$(2\theta_i - 1)\left[(1-2\gamma\theta_i) + 2(\gamma-1)\frac{\rho_i+\theta_i}{\tau+1} + \frac{2(\rho_m-\rho_i)+1-2\theta_i}{\tau+1+|R_i|}(N-|R_i|)\right] \leq 0.$$

If $\theta_i = 1$, we have:

$$\rho_m - \rho_i \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma-1)\frac{\tau-\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|},$$

which is trivially satisfied for $\rho_m \leq \rho_i$, but may be violated if $\rho_m > \rho_i$. If $\theta_i = 0$, we have:

$$\rho_i - \rho_m \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma-1)\frac{\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|},$$

which is trivially satisfied for $\rho_m \geq \rho_i$, but may be violated if $\rho_m < \rho_i$. Combining the two cases proves the proposition.

A.11.7 Proof of Proposition OA.5

Consider the equilibrium in which $\mu_o = \Theta_o$ and $\mu_p = \Theta_p$. Given these strategies, the manager's action as a function of messages μ_o, μ_p is

$$a_m(\mu_o,\mu_p) = b + \mu_o + \mu_p + \frac{\rho+\mu_o+\mu_p}{2\rho+S_o+S_p}(K-S_o-S_p). \tag{94}$$

First, consider the sender representing optimists. If he sends a truthful message, the manager's action will be $a_m(\Theta_o,\Theta_p)$, while if he sends a message $\mu \neq \Theta_o$, the manager's action will be $a_m(\mu,\Theta_p)$, where $a_m(\cdot,\cdot)$ is given by (94). The sender finds it optimal to communicate truthfully if and only if

$$\sum_{\theta_{-i}\in\{0,1\}^{K-S_o}}\left[\begin{array}{c}(b+\frac{\rho+\Theta_o+\Theta_p}{2\rho+S_o+S_p}(K-S_o-S_p)-\sum_{i=S_o+S_p+1}^{K}\theta_i)^2\\ -(b+\mu+\frac{\rho+\mu+\Theta_p}{2\rho+S_o+S_p}(K-S_o-S_p)-\Theta_o-\sum_{i=S_o+S_p+1}^{K}\theta_i)^2\end{array}\right]P_i(\theta_{-i}|\sum_{i=1}^{N_o}\theta_i=\Theta_o)\leq 0.$$

Simplifying this expression, we obtain

$$(\Theta_o-\mu)\frac{K+2\rho}{2\rho+S_o+S_p}\left[\begin{array}{c}2b+\frac{2\rho+\Theta_o+2S_p+\mu}{2\rho+S_o+S_p}(K-S_o-S_p)\\ +\mu-\Theta_o-2(K-S_o-S_p)\mathbb{E}_o\left[\theta_i|\sum_{i=1}^{N_o}\theta_i=\Theta_o\right]_i\end{array}\right]\leq 0$$

for all $\mu \in \{0,1,...,S_o\}$, which is equivalent to

$$(\Theta_o-\mu)\left[(2\rho+S_o+S_p)b-(K-S_o-S_p)\Delta-(\Theta_o-\mu)\left(\frac{K}{2}+\rho\right)\right]\leq 0.$$

Consider deviations by one signal (i.e., from a truthful message Θ_o to $\mu = \Theta_o - 1$ and $\mu =$

$\Theta_o + 1)$. A deviation to $\mu = \Theta_o - 1$ is not profitable if and only if

$$(2\rho + S_o + S_p)\, b - (K - S_o - S_p)\, \Delta \leq \rho + \frac{K}{2}.$$

A deviation to $\mu = \Theta_o + 1$ is not profitable if and only if

$$(K - S_o - S_p)\, \Delta - (2\rho + S_o + S_p)\, b \leq \frac{K}{2} + \rho.$$

Taken together, we have

$$|(2\rho + S_o + S_p)\, b - (K - S_o - S_p)\, \Delta| \leq \rho + \frac{K}{2}, \tag{95}$$

which coincides with the IC constraint in the basic model without communication among shareholders. We next show that if truthful reporting dominates sending $\mu = \Theta_o - 1$, then it dominates sending any $\mu < \Theta_o - 1$. To see this, note that

$$\begin{aligned} & (2\rho + S_o + S_p)\, b - (K - S_o - S_p)\, \Delta - (\Theta_o - \mu) \left(\frac{K}{2} + \rho \right) \\ < \ & (2\rho + S_o + S_p)\, b - (K - S_o - S_p)\, \Delta - \left(\frac{K}{2} + \rho \right) \leq 0, \end{aligned}$$

where the first inequality follows from the strict monotonicity of the expression in μ, and the second inequality is the condition that truthful reporting dominates sending $\mu = \Theta_o - 1$. Next, we show that if truthful reporting dominates sending $\mu = \Theta_o + 1$, then it dominates sending any $\mu > \Theta_o + 1$:

$$\begin{aligned} & (K - S_o - S_p)\, \Delta + (\Theta_o - \mu) \left(\frac{K}{2} + \rho \right) - (2\rho + S_o + S_p)\, b \\ < \ & (K - S_o - S_p)\, \Delta - \left(\frac{K}{2} + \rho \right) - (2\rho + S_o + S_p)\, b \leq 0, \end{aligned}$$

where the first inequality follows from the strict monotonicity of the expression in μ, and the second inequality is the condition that truthful reporting dominates $\mu = \Theta_o + 1$. Hence, (95) is both the necessary and the sufficient condition for truth-telling of the sender representing optimists (under the assumption that the sender representing pessimists reports Θ_p truthfully).

Second, consider the sender representing pessimists. By the argument identical to the argument for the sender representing optimists, this sender finds it optimal to report Θ_p truthfully if and only if

$$(\Theta_p - \mu) \left[(2\rho + S_o + S_p)\, b + (K - S_o - S_p)\, \Delta - (\Theta_o - \mu) \left(\frac{K}{2} + \rho \right) \right] \leq 0$$

for all $\mu \in \{0, 1, ..., S_p\}$. Notice that this inequality holds automatically for all $\mu > \Theta_p$, since the first multiple of the expression is negative and the second multiple is positive. Thus, it is sufficient to consider deviations to $\mu \leq \Theta_p - 1$. In this case, $\Theta_p - \mu > 0$, so the above inequality is equivalent to

$$(2\rho + S_o + S_p)\, b + (K - S_o - S_p)\, \Delta \leq (\Theta_o - \mu) \left(\frac{K}{2} + \rho \right).$$

Since the left-hand side does not depend on μ and the right-hand side is strictly decreasing in μ, it is necessary and sufficient to verify that the inequality holds for $\mu = \Theta_p - 1$, in which case:

$$(2\rho + S_o + S_p)\, b + (K - S_o - S_p)\, \Delta \leq \frac{K}{2} + \rho.$$

Notice that both inequalities are identical to the conditions for existence of the truth-telling equilibrium in the basic model.

A.11.8 Proof of Proposition OA.6

Consider shareholder i who is O_ith to send the message. Let $B_i \equiv R_i \cap \{j \in S, O_j < O_i\}$ be the set of shareholders that are expected to communicate truthfully who communicate before shareholder i. Similarly, let $A_i \equiv R_i \cap \{j \in S, O_j > O_i\}$ be the set of shareholders that are expected to communicate truthfully who communicate after shareholder i. By definition, $|R_i| = |B_i| + |A_i|$.

Consider the incentive constraint of shareholder i. Given the message of shareholder i and the belief that shareholders in set R_i communicate truthfully, the manager's action is given by (7) and (8) for truthful and non-truthful messages of shareholder i. Given that shareholder i already observes messages of shareholders in B_i, he has incentives to communicate truthfully if and only if

$$\sum_{\theta_{-i} \in \{0,1\}^{K-|B_i|-1}} \left[(a_m(\theta_{R_i}, \theta_i) - Z)^2 - (a_m(\theta_{R_i}, 1-\theta_i) - Z)^2 \right] P_i(\theta_{-(i,B_i)} | \theta_i, \theta_{B_i}) \leq 0, \qquad (96)$$

where $\theta_{-(i,B_i)}$ refers to the set of all signals that exclude the signal of shareholder i and the signals of shareholders in set B_i, and θ_{B_i} refers to the set of all signals of shareholders in set B_i. Plugging (7) and (8) into (96) gives

$$0 \geq \sum_{\theta_{-(i,B_i)}} \left[2\theta_i - 1 + (K - |R_i| - 1) \cdot \frac{2\theta_i - 1}{\tau + |R_i| + 1} \right]$$
$$\times \left[2b + (1 - 2\theta_i) - 2 \sum_{j \in -R_i \setminus \{i\}} \theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1} (K - |R_i| - 1) \right] P_i(\theta_{-(i,B_i)} | \theta_i, \theta_{B_i}).$$

Note that the first multiple in each term equals $(2\theta_i - 1)\frac{\tau + K}{\tau + |R_i| + 1}$. Thus, the above inequality is equivalent to

$$0 \geq (2\theta_i - 1) \sum_{\theta_{-(i,B_i)}} P_i(\theta_{-(i,B_i)}|\theta_i, \theta_{B_i}) \left(2b + (1 - 2\theta_i) - 2 \sum_{j \in -R_i \setminus \{i\}} \theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1} (K - |R_i| - 1) \right).$$

Since $\sum_{\theta_{-(i,B_i)}} \left(\sum_{j \in -R_i \setminus \{i\}} \theta_j \right) P_i(\theta_{-(i,B_i)}|\theta_i, \theta_{R_i}) = \frac{\rho_i + \mathbf{1}_{R_i} + \theta_i}{\tau + |R_i| + 1} (K - |R_i| - 1)$, we can further simplify it to

$$(2\theta_i - 1) \left[2b + (1 - 2\theta_i) + \frac{2(\rho_m - \rho_i) + 1 - 2\theta_i}{\tau + |R_i| + 1} (K - |R_i| - 1) \right] \leq 0.$$

Considering $\theta_i = 0$ and $\theta_i = 1$, we get the same IC constraint as in the base model, (10).

A.11.9 Derivations for the example in Section A.10

Consider the example in that section: $N = 6$, $N_o = 3$, $\rho = 1$, $\Delta = 0.5$, $b = 0.1$, $K = 10$, $\lambda = 0.2$, $u_0 = 5$.

It is straightforward to check that the constraint $(2\rho + N_o + 1)\, b + (K - N_o - 1)\, \Delta \leq \rho + \frac{K}{2}$ from Propositions 4 and 5 is satisfied for the assumed parameters. The proof of Proposition 4 shows that if all investors become shareholders, then all optimists and at least one pessimist communicate truthfully, and using (43) and (44),

$$\begin{aligned} \hat{r} &= \max_{|R| \in [4,6], |R| \in \mathbb{N}} \{|R| : (2 + |R|)\, 0.1 + (10 - |R|)\, 0.5 \leq 7\} = 6. \\ \hat{\lambda} &= \frac{0.2\,(10 - \hat{r})\, 3}{2 + \hat{r}} = 0.3. \end{aligned}$$

Hence $\lambda = 0.2 < \hat{\lambda}$, and the proof of Proposition 4 shows that in this case, only optimistic investors become shareholders in the baseline model. Since $\hat{r} = 6$, all three optimists communicate truthfully.

Thus, if there are no index funds ($L = 0$), $N_o = 3$ optimistic investors acquire shares of the firm ($\frac{1}{3}$ share each), communicate truthfully, and no pessimistic investor acquires shares and communicates. Using (12) and (50), the equilibrium valuations of optimists and pessimists and the stock price are:

$$\begin{aligned} \mathbb{E}_o[U|N_o] &= u_0 - b^2 + \tfrac{2b\Delta(K-N_o)}{2\rho+N_o} - \tfrac{\left(\rho^2-\Delta^2\right)(K-N_o)}{2\rho(2\rho+1)} \tfrac{2\rho+K}{2\rho+N_o} - \left[\tfrac{\Delta(K-N_o)}{2\rho+N_o}\right]^2 = 2.54, \\ \mathbb{E}_p[U|N_o] &= u_0 - b^2 - \tfrac{2b\Delta(K-N_o)}{2\rho+N_o} - \tfrac{\left(\rho^2-\Delta^2\right)(K-N_o)}{2\rho(2\rho+1)} \tfrac{2\rho+K}{2\rho+N_o} - \left[\tfrac{\Delta(K-N_o)}{2\rho+N_o}\right]^2 = 2.26, \\ p^* &= \mathbb{E}_o[U|N_o] - \tfrac{\lambda}{N_o} = 2.4733. \end{aligned} \tag{97}$$

Suppose now that each investor holds $\frac{\alpha}{N}$ shares at the start of the game. If no investor becomes an index fund, then the equilibrium valuations and share price are given by (97). Pessimistic investors sell their endowment, and optimistic investors buy additional shares from the original owners and pessimistic investors. Hence, the equilibrium payoff of a pessimistic investor (who sells his endowment) is $\frac{\alpha}{N}p^*$. Instead, suppose that one pessimistic investor announces that he will become an index fund, thereby committing to owning $\frac{1}{N} = \frac{1}{6}$ of the share. We conjecture, and later verify, that in the resulting equilibrium, pessimists' valuations are below the share price. Thus, the remaining $\frac{N-1}{N} = \frac{5}{6}$ shares will be held by the three optimistic investors in equal proportions ($\frac{5}{18}$ each). Since $\hat{r} = 6$, then in equilibrium, all four shareholders communicate to the manager, so $|R| = 4$. Using (12), the valuation of optimists and pessimists is

$$\mathbb{E}_o[U|N_o+1] = u_0 - b^2 + \tfrac{2b\Delta(K-N_o-1)}{2\rho+N_o+1} - \tfrac{\left(\rho^2-\Delta^2\right)(K-N_o-1)}{2\rho(2\rho+1)}\tfrac{2\rho+K}{2\rho+N_o+1} - \left[\tfrac{\Delta(K-N_o-1)}{2\rho+N_o+1}\right]^2 = 3.34,$$
$$\mathbb{E}_p[U|N_o+1] = u_0 - b^2 - \tfrac{2b\Delta(K-N_o-1)}{2\rho+N_o+1} - \tfrac{\left(\rho^2-\Delta^2\right)(K-N_o-1)}{2\rho(2\rho+1)}\tfrac{2\rho+K}{2\rho+N_o+1} - \left[\tfrac{\Delta(K-N_o-1)}{2\rho+N_o+1}\right]^2 = 3.14.$$

The equilibrium price $p_{L=1}$ is determined by the condition that supply equals demand:

$$3 \times \frac{\mathbb{E}_o[U|4] - p_{L=1}}{\lambda} = \frac{5}{6} \Leftrightarrow p_{L=1} = \mathbb{E}_o[U|4] - \frac{5}{18}\lambda = 3.2844.$$

Hence, indeed, the valuation of pessimists is below the price, consistent with the conjecture.

The utility of the pessimist if he commits to being an index fund is:

$$\frac{1}{N}\mathbb{E}_p[U|N_o+1] - \frac{1-\alpha}{N}p_{L=1} - \frac{\lambda}{2}\frac{1}{N^2}.$$

For the pessimistic investor to find it optimal to commit to being passive, we need this utility to exceed $\frac{\alpha}{N}p^*$, or equivalently,

$$\mathbb{E}_p[U|N_o+1] - (1-\alpha)\,p_{L=1} - \frac{\lambda}{2N} \geq \alpha p^* \Leftrightarrow$$
$$\alpha\left(p_{L=1} - p^*\right) \geq \frac{\lambda}{2N} - \mathbb{E}_p[U|N_o+1] + p_{L=1} \Leftrightarrow \alpha \geq \frac{0.1/6 - 3.14 + 3.2844}{3.2844 - 2.4733} = 0.1986.$$

Therefore, a pessimistic investor wants to commit to set up a passive fund if and only if $\alpha \geq 0.1986$.

At the same time, there is an upper bound on the number of investors that have incentives to commit to being passive. If another passive investor announces that he will become passive, the valuation of optimists and pessimists becomes

$$\mathbb{E}_o[U|N_o+2] = u_0 - b^2 + \tfrac{2b\Delta(K-N_o-2)}{2\rho+N_o+2} - \tfrac{\left(\rho^2-\Delta^2\right)(K-N_o-2)}{2\rho(2\rho+1)}\tfrac{2\rho+K}{2\rho+N_o+2} - \left[\tfrac{\Delta(K-N_o-2)}{2\rho+N_o+2}\right]^2 = 3.8624,$$
$$\mathbb{E}_p[U|N_o+2] = u_0 - b^2 - \tfrac{2b\Delta(K-N_o-2)}{2\rho+N_o+2} - \tfrac{\left(\rho^2-\Delta^2\right)(K-N_o-2)}{2\rho(2\rho+1)}\tfrac{2\rho+K}{2\rho+N_o+2} - \left[\tfrac{\Delta(K-N_o-2)}{2\rho+N_o+2}\right]^2 = 3.7196,$$

and the equilibrium price increases to

$$p_{L=2} = \mathbb{E}_o[U|5] - \frac{2}{9}\lambda = 3.818.$$

If a pessimistic investor deviates and becomes active, he expects to sell his shares and get price $\frac{\alpha}{N}p_{L=1}$ (since there will be one pessimistic index fund). Hence, the investor finds it optimal not to deviate if and only if

$$\frac{1}{N}\mathbb{E}_p[U|N_o+2]-\frac{1-\alpha}{N}p_{L=2}-\frac{\lambda}{2}\frac{1}{N^2}\geq\frac{\alpha}{N}p_{L=1}\Leftrightarrow$$

$$\alpha\left(p_{L=2}-p_{L=1}\right)\geq\frac{\lambda}{2N}-\mathbb{E}_p[U|N_o+2]+p_{L=2}\Leftrightarrow\alpha\geq\frac{0.1/6-3.7196+3.818}{3.818-3.2844}=0.2157.$$

We conclude that if $\alpha\in[0.1986,0.2157)$, then exactly one pessimistic shareholder commits to becoming a passive investor.