# On the Routing Capacity Regions of Networks

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**Abstract**

In this article we study a new characterization of routing capacity region of networks. We begin by extending the famous "Japanese theorem" on the characterization of the routing capacity region of networks with multiple unicast sessions in terms of an infinite set of linear inequalities to the multiple multicast networks. Then we study a systematic approach to eliminate the redundant inequalities and forming a minimal set of necessary and sufficient inequalities. Using this technique, we find the capacity of undirected ring networks for different scenarios. Then we discuss new results on the complexity of the polytope of routing region of general networks and derive tight upper and lower bounds in terms of the size of the underlying graph of the network. A multitude of tools from graph theory, theory of integer programming, geometry, and probability are used in deriving the results.

## 1 Introduction

Classical communication theory is based on the notion of routing. In routing, messages are treated as commodities and transmission protocols are based on the flow of commodities over the network.

Recently, with the newer paradigm of network coding (Ahlswede et al., 2000), our understanding of the optimal transmission schemes over communication networks has been revolutionized. In network coding, messages in the network are treated as mathematical objects and the network nodes are allowed to perform mathematical operations on the content of the packets of data to generate new packets. A simple butterfly example in the pioneering work of Ahlswede, et al. (Ahlswede et al., 2000), demonstrated that there are networks for which the latter approach is necessary to achieve the highest throughput of the network.

Even with the advent of network coding, the backbone of the modern information technologies for transmission over internet or cell networks is based on the routing. The reason is twofold. First, while routing in general does not achieve capacity and is suboptimal, the computational complexity of the nodes that perform routing is substantially lower than the nodes that perform optimal network coding protocols. Since, complexity and power are major concerns in modern communication designs, routing is still the most popular approach in this regard. The second fold is the theoretical support of the performance of routing schemes, compared to network coding schemes. In a breakthrough work (Leighton & Rao, 1999), Leighton and Rao proved that routing can achieve rates that are at least a log factor away from the min-cut value of the network. Since, network coding rates are bounded by the min-cut of the network, routing is at most a log factor less than the network coding capacity. This result shows that by restricting to routing strategies, the loss in the capacity is not substantial.

Traditionally, routing is applied to wired networks. Recently, Avestimehr, Diggavi and Tse (Avestimehr et al., 2011), proposed a simple linear model for wireless networks that approximates the capacity of wireless networks within a constant gap. Using the linear model, a number of flow based transmission schemes has been proposed for wireless networks. The more recent paper (Chekuri et al., 2012) extends the results on suboptimality of routing for wired networks (Leighton & Rao, 1999) to wireless networks. These results extend the applicability of routing strategies to a broader setting of wired and wireless networks.

In this article we focus on the routing capacity of networks. Much of the routing literature focuses on the *multicommodity flow* problem in which every message in the network is transmitted from a source to a unique destination ((Stoenescu et al., 2007a),(Stoenescu et al., 2007b)). The famous *max-flow min-cut* theorem provides bounds on the rates of the different messages being simultaneously transmitted between the different source-destination pairs. (Schrijver, 2003, Part VII) surveys many of the cases where this bound is tight. The paper (Hu, 1963) is an early reference which provides an example where the bound is not tight.

The papers (Iri, 1971) and (Onaga, 1970) establish a result sometimes called the “Japanese theorem”, a special case of Farkas’ lemma, which provides necessary and sufficient conditions for determining if an arbitrary set of rates is feasible in a network. One shortcoming of this result is that the description of the routing capacity region for the multicommodity flow problem involves the intersection of an infinite set of inequalities. Another shortcoming of this result is that it only considers unicast sessions. While the assumption of a unique destination is natural for many application areas of network optimization, for communication problems we want to allow for the possibility of messages from a single transmitter to multiple receivers. Using standard terminology from communications, we further refer to *unicast* or *multicast* messages to indicate if the set of destinations is a single terminal or a set of multiple terminals. We will use the terms unicast and multicommodity flow interchangeably.

Just as one can form a system of linear inequalities to describe a multicommodity flow problem, one can likewise study the general multiple multicast problem where every terminal in the network potentially has messages for every non-empty subset of the other accessible terminals. For a multicommodity flow or unicast session the flow for a session which enters an intermediate vertex along the path is identical to the flow for that session emanating from that vertex. The natural generalization for multicast sessions constrains each spanning subtree carrying flow to have all of the edges of that subtree transmit the same flow. The set of flows along the various paths and subtrees are jointly constrained by the capacities of the edges or nodes in the graph, and the corresponding fractional routing capacity region can in principle be determined by Fourier-Motzkin elimination (Schrijver, 1998). However, as the results of Fourier-Motzkin elimination are specific to the set of constraints for a particular networking problem, our objective is to offer a characterization which will apply to many networking problems.

In this article we offer a systematic approach to reducing the infinite set of inequalities that describe the multiple multicast capacity of an arbitrary network into a finite set of inequalities. We begin by extending the Japanese theorem from multiple unicase networks to multiple multicast networks. We then observe a consequence of our reduction method on characterizing the multiple multicast capacity of ring networks. Our technique determines the minimal necessary and sufficient inequalities that describe the capacity region.

Next we focus on the *size* of the coefficients of the inequalities that appear in the minimal description of the routing rate region of general undirected networks. We combine the inequality elimination technique with complexity results (see, e.g., (Schrijver, 1998), (Grötschel et al., 1988)) on the description of a rational system of linear inequalities to bound the coefficients of the linear inequalities that describe the routing rate region. The obtained bounds are exponential in the number of edges of network. We further discuss an average case analysis of the size of linear inequalities for undirected ring networks, and by applying a probabilistic technique we suggest that for the characterization of routing capacity region we truly need to take into account the inequalities with the coefficients that grow polynomially in the number of edges of network.

# 2 The Routing Capacity Region of General Networks

## 2.1 Preliminaries

Consider a network that is represented by a graph $G(V,E)$, where $V$ and $E$ respectively denote the set of vertices and edges in the network graph. The edges are either all undirected, meaning that the sum of flow along both directions of an edge is bounded by the capacity of the edge, or all directed. Furthermore, for any subgraph $S$ of the network let $V(S)$ and $E(S)$ respectively denote its set of vertices and edges of the subgraph $S$. In a general communication setting, every vertex $v \in V$ can simultaneously send messages to arbitrary nonempty subsets of accessible vertices in $V \setminus \{v\}$. Every message $M$ with source node $v_s$ and set of destination nodes $\{v_1, \cdots, v_k\}$, is associated with a rate $R_M$ and with a set of $t(M)$ spanning subtrees, $\{T_M^1, \cdots, T_M^{t(M)}\}$, that connect $v_s$ to $\{v_1, \cdots, v_k\}$. We assume throughout that rates from any source to a set of vertices that include an inaccessible destination are always set to zero.

For message $M$, let $r_M^j$ be the amount of flow for that message that passes through spanning subtree $T_M^j$, $j \in \{1, \ldots, t(M)\}$. We then have

$$\sum_{j=1}^{t(M)} r_M^j = R_M.$$

Flows of the network satisfy the capacity constraints

$$\sum_{M,j:\, e \in E(T_M^j)} r_M^j \leq C_e,\ e \in E,$$

where $C_e$ denotes the capacity of edge $e$.

A rate-tuple $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ corresponding to the sessions $M_1, \cdots, M_N$ is said to be *feasible* if for each $i \in \{1, \ldots, N\}$ there exists an assignment of $\{r_{M_i}^1, \cdots, r_{M_i}^{t(M_i)}\}$ such that

$$\sum_{j=1}^{t(M_i)} r_{M_i}^j = R_{M_i}$$

and the edge constraints are fulfilled. Our goal is to offer a new way of thinking about the set of feasible rate-tuples in a given network and to provide new characterizations of routing capacity regions.

## 2.2 Generalizations of the Japanese Theorem

The Japanese theorem characterizes the set of feasible routing rates-tuples for edge-constrained networks in problems where there are only multiple unicast sessions in terms of an infinite set of inequalities. Each inequality is in terms of a "distances function". A distance function is a function that assigns a positive integer to each edge in the network which is called the "distance" of the edge. For each inequality we further need to find the shortest path lengths for each session with respect to the corresponding distance function. The length of a path is the sum of distances of the edges on that path and the shortest path is the path with the shortest length. In what follows, $\mathbb{Z}^+$ refers to the set of nonnegative integers.

**Theorem 1** (Generalized Japanese theorem for edge-constrained networks (Yazdi et al., 2010))**.** *Consider a directed or an undirected edge-constrained network $G(V,E)$. For function $f : E \to \mathbb{Z}^+$, define $L_f(T) = \sum_{e \in E(T)} f(e)$*

*and* $\ell_f(M) = \min_{j\in\{1,\ \dots,\ t(M)\}} L_f(T_M^j)$*. The rate-tuple* $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ *is feasible in* $G(V,E)$ *if and only if for every function* $f : E \to \mathbb{Z}^+$*, the following inequality holds:*

$$\sum_{i=1}^{N} \ell_f(M_i) R_{M_i} \leq \sum_{e\in E} f(e) C_e. \tag{1}$$

*Proof.* A rate vector $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ is routable in a network $G(V,E)$ if and only if the following linear program has a solution for $\{r_T\}$:

1. $\sum_{j=1}^{t(M_i)} r_{M_i}^j \geq R_{M_i}$ for every $i \in \{1, \cdots, N\}$
2. $\sum_{M,j:e\in E(T_M^j)} r_M^j \leq C_e$ for every $e \in E$.
3. $0 \leq r_M^j$, for every $M$ and $1 \leq j \leq t(M)$.

Notice that the first set of inequalities can be changed to equalities, but it is equivalent and more convenient here to work with inequalities.

Label the elements of $E$ from 1 to $|E|$. Let

$$\delta_{e,T_M^j} = \begin{cases} 1, & e \in E(T_M^j) \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\mathbf{r} = \left(r_{M_1}^1, \cdots, r_{M_1}^{t(M_1)}, \cdots, r_{M_N}^1, \cdots, r_{M_N}^{t(M_N)}\right)^T \tag{2}$$

$$\mathbf{c} = \left(C_1, \cdots, C_{|E|}, -R_1, \cdots, -R_{M_N}, 0, \cdots, 0\right)^T \tag{3}$$

and matrix $\mathbf{M}$ as follows:

$$\mathbf{M} = \left(\begin{array}{ccc|ccc|ccc}
\delta_{1,T_{M_1}^1} & \dots & \delta_{1,T_{M_1}^{t(M_1)}} & & & & \delta_{1,T_{M_N}^1} & \dots & \delta_{1,T_{M_N}^{t(M_N)}} \\
\vdots & \ddots & \vdots & & \dots & & \vdots & \ddots & \vdots \\
\delta_{|E|,T_1^1} & \dots & \delta_{|E|,T_1^{t(M_1)}} & & & & \delta_{|E|,T_{M_N}^1} & \dots & \delta_{|E|,T_{M_N}^{t(M_N)}} \\
\hline
-1 & \dots & -1 & & & & 0 & \dots & 0 \\
0 & \dots & 0 & & \dots & & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & & & & 0 & \dots & 0 \\
0 & \dots & 0 & & & & -1 & \dots & -1 \\
\hline
-1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
 & \ddots & & & & & & & \\
 & & -1 & & & & & & \\
 & & & -1 & & & & & \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
 & & & & & -1 & & & \\
 & & & & & & -1 & & \\
 & & & & & & & \ddots & \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & -1
\end{array}\right). \tag{4}$$

Then a routing-feasible assignment of $\{r_M^j\}$ satisfies the following matrix inequality:

$$\mathbf{Mr} \leq \mathbf{c}. \tag{5}$$

Farkas' lemma (see, e.g., (Ziegler, 1995, §1.4)) provides necessary and sufficient conditions for the feasibility of a system of linear inequalities. The following lemma applies Farkas' lemma to inequality (5):

**Lemma 2** (Farkas)**.** *There exists a solution to* (5) *if and only if every row vector* $\boldsymbol{v}^T$ *with* $\boldsymbol{v} \geq \boldsymbol{0}$ *and* $\boldsymbol{v}^T\boldsymbol{M} = \boldsymbol{0}$ *satisfies* $\boldsymbol{v}^T\boldsymbol{c} \geq 0$.

We define

$$\mathbf{v}^T = \left(v_1, \cdots, v_{|E|}, v_{|E|+1}, \cdots, v_{|E|+N}, v_{|E|+N+1}, \cdots, v_{|E|+N+\sum_M t(M)}\right). \tag{6}$$

Note that the steps of Fourier-Motzkin elimination maintain rational or integral rate coefficients throughout the procedure; this is why we need not consider irrational edge distances. Equation $\mathbf{v}^T\mathbf{M} = \mathbf{0}$ implies:

$$\sum_{e\in E} v_e \delta_{e,T_{M_i}^j} - v_{|E|+i} - v_{|E|+N+z_i^j} = 0,\ i \in \{1,\cdots,N\}, j \in \{1,\cdots,t(M_i)\}, \tag{7}$$

where

$$z_i^j = \begin{cases} t(M_1)+\cdots+t(M_{i-1})+j, & i \neq 1 \\ j, & i = 1 \end{cases}$$

Therefore by Lemma 2, for every $\mathbf{v} \geq \mathbf{0}$ satisfying (7), the inequality $\mathbf{v}^T\mathbf{c} \geq 0$ must hold. It can be written as

$$v_{|E|+1}R_{M_1} + v_{|E|+2}R_{M_2} + \cdots + v_{|E|+N}R_{M_N} \leq \sum_{e\in E} v_e C_e. \tag{8}$$

Fix a distance function $f = (f_1, \cdots, f_{|E|})$ and let

$$\mathbf{v}_f = \{\mathbf{v} \geq \mathbf{0} : (v_1, \cdots, v_{|E|}) = (f_1, \cdots, f_{|E|})\}.$$

Then for $\mathbf{v} \in \mathbf{v}_f$ inequality (7) can be written as

$$L_f(T_{M_i}^j) - v_{|E|+i} - v_{|E|+N+z_i^j} = 0,\ i \in \{1,\cdots,N\}, j \in \{1,\cdots,t(M_i)\}, \tag{9}$$

and inequality (8) can be written as

$$v_{|E|+1}R_{M_1} + v_{|E|+2}R_{M_2} + \cdots + v_{|E|+N}R_{M_N} \leq \sum_{e\in E} a_e C_e. \tag{10}$$

Since $v_{|E|+N+z_i^j} \geq 0$, then by (9), $v_{|E|+i} \leq L_f(T_{M_i}^j)$ for every $i$ and $j$. Therefore for every $\mathbf{v} \in \mathbf{v}_f$, $v_{|E|+i}$ can be bounded from above by $\min_j L_f(T_{M_i}^j) = \ell_f(M_i)$. Observe that it is possible to choose $v_{|E|+i} = \ell_f(M_i)$ for every $i$ by setting $v_{|E|+i} = \ell_f(M_i)$ and $v_{|E|+N+z_i^j} = L_f(T_{M_i}^j) - \ell_f(M_i)$. Next notice that the left hand side of (10) is maximized among vectors in $\mathbf{v}_f$ when the values of $v_{|E|+i}$ are maximized; i.e., when $v_{|E|+i} = \ell_f(M_i)$. Equivalently (10) holds if and only if the following inequality is satisfied:

$$\sum_M \ell_f(M) R_M \leq \sum_{e\in E} a_e C_e; \tag{11}$$

this is the routing bound corresponding to the distance function $f$. □

To give an intuitive explanation of the Japanese theorem, consider a single unicast session in an edge constrained network. Then the famous max–flow min–cut theorem of Ford and Fulkerson provides the capacity of the transmission from the source node to the destination node. It is easy to verify that max–flow min–cut theorem is equivalent to considering the constraints of the form (1) when the distance functions are restricted to the functions with the range $\{0,1\}$ and each such function corresponds to an edge cut that separates the source node from the destination node. Each such function assigns value one to all edges that form a cut from the source node to the destination node and all other edges will be assigned a zero value. The Japanaese theorem states that in the case that all sessions are present in the network, the capacity of the network is characterized by all possible distance functions with the range of positive integers.

### 2.3 An Inequality Elimination Technique

Theorem 1 is unsatisfying because it describes the routing capacity region with infinitely many inequalities, and by Fourier-Motzkin elimination we know that the collection of feasible rate-tuples is a polytope defined by a finite set of inequalities. To circumvent this issue, we establish when a Japanese theorem inequality is redundant for a networking problem by exploiting the special structure of these inequalities. As we will see later, this approach enables us to offer a description of the set of feasible rate-tuples for the multiple multicast problem with only finitely many inequalities.

Every inequality in (1) is a description of a halfspace in the space of rate-tuples $\mathbb{R}^N$, and the feasible polytope of rate-tuples is the intersection of these half spaces with the half spaces corresponding to the non-negativity of rates. The boundary points on the feasible set of rates belong to the hyperplanes defined by:

$$\sum_{i=1}^{N} \ell_f(M_i) R_{M_i} = \sum_{e\in E} f(e) C_e, \tag{12}$$

The following result provides the necessary and sufficient conditions for a *feasible* rate-tuple to be on a hyperplane of the form (12).

**Theorem 3** ((Yazdi et al., 2010)). *Consider a directed network or an undirected network and fix any distance function $f$. The feasible rate tuple $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ is on the hyperplane (12) if and only if there exists a feasible assignment $\{r^j_{M_i} : i \in \{1,\cdots,N\}, j \in \{1,\cdots,t(M_i)\}\}$ with the properties:*

1. *$r^j_{M_i} = 0$ if $L_f(T^j_{M_i}) > \ell_f(M_i)$, and for the appropriate setting*
2. *(Edge-constrained setting) $\sum_{\{i,j:\, e\in E(T^j_{M_i})\}} r^j_{M_i} = C_e$ for $f(e) > 0$*

*Proof.* To establish the necessity of Conditions 1 and 2 in Theorem 3, suppose that the rate-tuple $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ is feasible and is on the defining hyperplane (12) corresponding to function $f$. The sum of the flows passing through any edge $e$ in the network at most $C_e$. By multiplying both sides of this constraint by $f(e)$ and adding up the resulting inequalities over all edges in the network we find that

$$\sum_{i=1}^{N} \sum_{j=1}^{t(M_i)} L_f(T^j_{M_i}) r^j_{M_i} \le \sum_{e\in E} f(e) C_e. \tag{13}$$

A lower bound for the left-hand side of the preceding inequality is obtained when all sessions are routed along their shortest subtrees with respect to $f$:

$$\sum_{i=1}^{N} \ell_f(M_i) R_{M_i} \le \sum_{i=1}^{N} \sum_{j=1}^{t(M_i)} L_f(T^j_{M_i}) r^j_{M_i} \le \sum_{e\in E} f(e) C_e. \tag{14}$$

Since the rate-tuple is on the hyperplane given by (12) by assumption, it follows that Condition 1 holds. To arrive at a contradiction, suppose next that Condition 2 is invalid. Hence the rate-tuple $\mathcal{R}$ is also feasible in another network with edge capacities $C'_e$, $e \in E$, in which $C'_e \leq C_e$ for all $e$ with strict inequality for at least one value of $e$ with $f(e) > 0$. Then Theorem 1 implies that

$$\sum_{i=1}^{N} \ell_f(M_i) R_{M_i} \leq \sum_{e \in E} f(e) C'_e < \sum_{e \in E} f(e) C_e, \tag{15}$$

which contradicts (12). Thus Condition 2 holds.

To establish sufficiency, consider a feasible rate-tuple which satisfies Conditions 1 and 2. The argument for constraint (13) applies for any feasible point, and Condition 2 implies that (13) can be replaced by the equality $\sum_{i=1}^{N} \sum_{j=1}^{t(M_i)} L_f(T^j_{M_i}) r^j_{M_i} = \sum_{e \in E} f(e) C_e$. By Condition 1 we further know that

$$\sum_{i=1}^{N} \ell_f(M_i) R_{M_i} = \sum_{i=1}^{N} \sum_{j=1}^{t(M_i)} L_f(T^j_{M_i}) r^j_{M_i} = \sum_{e \in E} f(e) C_e. \tag{16}$$

Hence, the rate-tuple $\mathcal{R}$ is on the defining hyperplane corresponding to function $f$, completing the proof. $\square$

We focus here on distance functions $f$ that result in *nontrivial* inequalities with $\ell_f(M_i) > 0$ for at least one $i \in \{1, \ldots, N\}$. We say that a nontrivial Japanese theorem inequality is *redundant* if for any assignment of capacities the *feasible* rate-tuples on the corresponding defining hyperplane all lie on the hyperplane bounding another nontrivial Japanese theorem inequality. We will next demonstrate that Theorem 3 implies a way to establish whether or not an inequality coming from the Japanese theorem or its extensions is redundant for a given networking problem. In the next section we will show that our inequality elimination technique enables us to characterize the fractional routing capacity region for the multiple multicast problem with a finite number of inequalities. As we will see, the true significance of the distance function is summarized by the collections of shortest paths for the unicast sessions and shortest subtrees for the multicast sessions corresponding to that function.

**Theorem 4** (Elimination Theorem (Yazdi et al., 2010))**.** *Given an edge-constrained network with a set of messages $\{M_1, \cdots, M_N\}$, consider two nontrivial distance functions $f$ and $g$ which are not identical. The network may be either directed or undirected. If*

1. *for every $e \in E$, $f(e) = 0$ whenever $g(e) = 0$, and*
2. *for every session $M_i, i \in \{1, \cdots, N\}$ and for all $j \in \{1, \cdots, t(M_i)\}$ the property $L_g(T^j_{M_i}) = \ell_g(M_i)$ implies $L_f(T^j_{M_i}) = \ell_f(M_i)$ (but not necessarily the converse),*

*then the half space* (1) *corresponding to $g$ is redundant in the description of the fractional routing capacity region given the half space corresponding to $f$.*

Before we prove this result, we will illustrate it with an example. Consider an undirected triangle network with $V = \{1, 2, 3\}$ and suppose $C_{(1,2)} = C_{(2,3)} = C_{(3,1)} = 1$. We permit all possible unicast and multicast sessions. Take $g((1,2)) = 2$, $g((2,3)) = 1$, and $g((3,1)) = 3$. It is easy to verify

- $\ell_g(1 \to 2) = \ell_g(2 \to 1) = 2$ and the shortest path is $(1,2)$,
- $\ell_g(2 \to 3) = \ell_g(3 \to 2) = 1$ and the shortest path is $(2,3)$,
- $\ell_g(3 \to 1) = \ell_g(1 \to 3) = 3$ and both paths are shortest, and

- $\ell_g(1 \to \{2,3\}) = \ell_g(2 \to \{1,3\}) = \ell_g(3 \to \{1,2\}) = 3$ and the shortest subtree is $\{(1,2),(2,3)\}$.

Therefore, the half space corresponding to distance function $g$ is

$$2(R_{1\to 2}+R_{2\to 1})+(R_{2\to 3}+R_{3\to 2})+3(R_{3\to 1}+R_{1\to 3})+3(R_{1\to\{2,3\}}+R_{2\to\{1,3\}}+R_{3\to\{1,2\}}) \\ \leq 2C_{(1,2)}+C_{(2,3)}+3C_{(3,1)}=6. \tag{17}$$

Next take $f((1,2)) = 1$, $f((2,3)) = 0$, and $f((3,1)) = 1$. Notice that the shortest paths and shortest subtrees for each session under distance function $g$ remain shortest paths and shortest subtrees for the sessions under $f$, although $f$ has a second shortest path for unicast sessions $1 \to 2$ and $2 \to 1$ and a second shortest subtree for the multicast sessions. The halfspace corresponding to $f$ is

$$(R_{1\to 2}+R_{2\to 1})+(R_{3\to 1}+R_{1\to 3})+(R_{1\to\{2,3\}}+R_{2\to\{1,3\}}+R_{3\to\{1,2\}}) \leq C_{(1,2)}+C_{(3,1)}=2. \tag{18}$$

The theorem states that (17) is redundant for defining the routing capacity region in the presence of (18). The reason is that a polytope is defined by a collection of hyperplanes, and every *feasible* rate-tuple like $R_{1\to 2} = R_{2\to 3} = R_{3\to 1} = 1$, $R_M = 0$, $M \notin \{1 \to 2,\ 2 \to 3,\ 3 \to 1\}$ which satisfies (17) with equality must also satisfy (18) with equality. The rate-tuple $R_{1\to\{2,3\}} = 2$, $R_M = 0$, $M \notin \{1 \to \{2,3\}\}$ is an example of an *infeasible* rate-tuple which satisfies (17) with equality; it is infeasible because four units of capacity are needed to transmit two units of multicast traffic, and the network has only three units of capacity. For the problem of characterizing the routing capacity region of a network we can ignore the infeasible rate-tuples.

We assert here that there are generalizations of Theorem 4 to some other classes of systems of linear inequalities. To save space we only give the proof for the edge-constrained setting from (Yazdi et al., 2010). (Yazdi et al., 2010) offers an alternate proof of the same result which uses the formalism of linear algebra.

*Proof.* Consider a feasible rate-tuple on the hyperplane (1) corresponding to $g$. By Condition 1 of Theorem 3, every session is routed only along the shortest paths and shortest subtrees associated with $g$, and hence by assumption only along the shortest paths and shortest subtrees corresponding to $f$. Furthermore, note that by assumption any edge $e$ with $f(e) > 0$ implies $g(e) > 0$ and so Condition 2 of Theorem 3 implies that this edge must be fully utilized. By Theorem 3 it follows that the feasible rate-tuple is also on the hyperplane (1) corresponding to $f$. Since the routing capacity region can be described in terms of its defining hyperplanes, the bound corresponding to $g$ is redundant given the inequality corresponding to $f$. □

Let $G(V,E)$ denote a ring network with the set of nodes $V = \{1,\cdots,n\}$. Let $E = \{1,\cdots,n\}$ where edge $i$ connects node $i$ and $i+1$ together. The paper (Yazdi et al., 2010) considers two different scenarios of multiple multicast sessions on undirected ring networks:

1. The sessions are either unicast which means that every node can send a message to any other node in the network, or broadcast which means that every node can send a message to all other nodes in the network.
2. A node can only send messages to subsets of nodes that form a line on the ring. For instance node 1 can send a message to nodes $n$ and 2 or another message to nodes $2,3$ and 4 since they form lines on the ring.

For these two cases the routing rate region is characterized by the following theorem:

**Theorem 5** ((Yazdi et al., 2010))**.** *For the two above scenarios, the routing rate region is characterized by the set of all distance functions with zero and one values.*

The proof of Theorem 5 is based on a geometric arguments that provides an algorithm for reducing any given distance function to a zero-one distance function while preserving the properties of Elimination Theorem.

We briefly present the reduction algorithm for each scenario. The proof of the performance of the algorithm in each case appears in (Yazdi et al., 2010). Let $g$ be an arbitrary distance function and $f$ be the corresponding zero-one distance function for which the properties of the Elimination Theorem holds.

### Algorithm for the first case

1. If all $g(e)$ are equal, then set $f(e) = 1$ for all edges. Otherwise, proceed to the next step.
2. Draw a circle $\mathcal{C}$, with points on its perimeter corresponding to each vertex of the ring so that the length of the arc between two adjacent points on the circle is proportional to the corresponding edge distance in $g$.
3. From each point on the perimeter of $\mathcal{C}$ draw a diameter originating from that point.
4. If the arc corresponding to an edge on $\mathcal{C}$ intersects at least one diameter, then set the corresponding edge distance in $f$ to one; otherwise set it to zero.

### Algorithm for the second case

1. Set $f(e) = 0$ for all $e$ with $g(e) = 0$.
2. Set $i = 1$.
3. Complete the following steps:
   (a) Set $f(i) = 1$.
   (b) Search for an index $j$ such that $\sum_{e=i+1}^{j-1} g(e) < \max_e g(e)$, but $\sum_{e=i+1}^{j} g(e) \geq \max_e g(e)$. If such a $j$ exists, it must be unique. In this case let $i = j$ and return to Step 3. If no such $j$ exists, go on to Step 4.
4. Let $f(e) = 0$ for the remaining edges.

Although we were able to characterize the exact routing capacity region for two special cases, it appears difficult to apply our tools to arbitrary collections of multiple multicast sessions and/or arbitrary networks. However, these ideas offer some insights that help to further characterize the minimal set of distance functions that define routing capacity region for general multiple multicast networks.

Finally, we remark that deriving tight network coding bounds on the capacity of cyclic graphs is in general very difficult. For the two cases of our study, using the simple characterization of the routing rate region, paper (Yazdi et al., 2010) concentrates on providing network coding bounds corresponding to zero one distance functions and show that these bounds are also valid in the network coding setting. This shows that routing is the optimal transmission protocol for the two cases of study and network coding does not enlarge the capacity region.

We next present other consequences of the inequality elimination theorem.

## 3 On the Complexity of the Routing Capacity Region

In previous section we observed that for special cases of multiple multicast networks, only zero-one distance function are sufficient for describing the routing rate region. However, for general multiple multicast problem in ring networks higher values of distance functions are necessary for describing the routing rate region:

**Theorem 6** ((Yazdi et al., 2010)). *Consider an undirected ring $G(V,E)$ with node set $V = \{1, \cdots, n\}$ and edge set $\{1, \cdots, n\}$, $n \geq 5$, such that edge $i$ connects nodes $i$ and $i+1$. If $G$ supports all multicast sessions, the distance function $g = (x, 1, \cdots, 1)$ for $2 \leq x \leq \lfloor \frac{n-2}{2} \rfloor$ can not be reduced to any function $f = (f_1, \cdots, f_n)$ with* $\max\{f_1, \cdots, f_n\} < x$.

*Proof.* To arrive at a contradiction, assume that we have found a $f$ with $\max\{f_1,\cdots,f_n\} < x$ that satisfies the conditions of Theorem 4. Let $M$ be an arbitrary multicast session with source node $o$ and destination nodes $\{d_1,d_2,\cdots,d_K\}$, where $d_1 < \cdots < d_i < o < d_{i+1} < \cdots < d_K$. Note that the complements of trees for session $M$ is

$$\{G(d_1,d_2),\cdots,G(d_i,o),G(o,d_{i+1}),G(d_{i+1},d_{i+2}),\cdots,G(d_{K-1},d_K),G(d_K,d_1)\},$$

where $G(i,j)$ denotes part of the graph $G$ between nodes $i$ and $j$ in clockwise direction. Thus to satisfy the conditions of Theorem 4, the longest complementary trees with respect to $g$ should remain longest under $f$.

Consider the multicast session from 1 to $\{2,x+2,x+3,\cdots,n\}$. Here among the complementary trees $G(1,2),G(2,x+2),\cdots,G(n,1)$ there are two longest trees $G(1,2)$ and $G(2,x+2)$ under $g$, and hence they should remain longest under $f$. Thus $f_1 = \sum_{i=2}^{x+1} f_i$ . Likewise consider the multicast sessions from 1 to $\{2,3,x+3,\cdots,n\}$, from 1 to $\{2,3,4,x+4,\cdots,n\}$, $\cdots$, from 1 to $\{2,3,\cdots,n-x,n\}$, and from 1 to $\{2,3,\cdots,n-x+1\}$ to obtain the constraints:

$$\begin{aligned} f_1 &= f_2+f_3+\cdots+f_{x+1} \\ f_1 &= f_3+f_4+\cdots+f_{x+2} \\ &\vdots \\ f_1 &= f_{n-x+1}+f_{n-x+2}+\cdots+f_n. \end{aligned} \tag{19}$$

Therefore,

$$f_2 = f_{x+2},\ f_3 = f_{x+3},\ \cdots,\ f_{n-x} = f_n. \tag{20}$$

Next consider the multicast sessions from 1 to $\{3,x+4,\cdots,n\}$, from 1 to $\{3,4,x+5,\cdots,n\}$, $\cdots$, from 1 to $\{3,4,\cdots,n-x-1,n\}$, and from 1 to $\{3,4,\cdots,n-x\}$. The constraints maintaining the longest complementary trees with respect to $g$ results in the following set of equalities:

$$\begin{aligned} f_1+f_2 &= f_3+f_4+\cdots+f_{x+3} \\ f_1+f_2 &= f_4+f_5+\cdots+f_{x+4} \\ &\vdots \\ f_1+f_2 &= f_{n-x}+f_{n-x+1}+\cdots+f_n \end{aligned} \tag{21}$$

Hence,

$$f_3 = f_{x+4},\ f_4 = f_{x+5},\ \cdots,\ f_{n-x-1} = f_n. \tag{22}$$

Since $n-x \geq x+2$, (20) and (22) imply:

$$f_2 = f_3 \cdots = f_n \doteq f_0 \tag{23}$$

and

$$f_1 = x f_0. \tag{24}$$

Since distance function $f \neq \mathbf{0}$, it follows that $f_0 \neq 0$. Hence $f_1 = x f_0$ should be an integer bounded below by $x$, which is a contradiction. □

Theorem 6 demonstrates that the maximum value of the functions in the minimal set of distance functions that describe the routing rate region, grow at least linearly with the size of the network. From a computational perspective, it is important to understand the behavior of the maximum value of the minimal set of distance functions that describe routing rate region of a graph. In (Kakhbod & Yazdi, 2010) authors use tools from the theory of polytopes and solutions to integer programs to derive such bounds.

Let $p$ and $q$ be relatively prime integers and let $\alpha = p/q$. Define

$$size(\alpha) = 1 + \lceil \log_2(1+|p|) \rceil + \lceil \log_2(1+|p|) \rceil .$$

For the rational vector $\mathbf{c} = (\gamma_1, \cdots, \gamma_n)$ and the rational matrix $A = (\alpha_{i,j})_{1 \le i \le m, 1 \le j \le n}$ we have:

$$size(\mathbf{c}) = n + size(\gamma_1) + \cdots + size(\gamma_n) \tag{25}$$

$$size(A) = mn + \sum_{m,n} size(\alpha_{i,j}) \tag{26}$$

Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$. Then the size of the linear inequality $\mathbf{ax} \le \alpha$ is defined as $1 + size(\mathbf{a}) + size(\alpha)$. The size of a system $A\mathbf{x} \le \mathbf{b}$ of linear inequalities is defined as $1 + size(A) + size(\mathbf{b})$. Next let $P \subset \mathcal{R}^n$ be a rational polyhedron. The *facet complexity* of $P$ defined as the smallest number $\Lambda \ge n$ for which there exists a system $A\mathbf{x} \le \mathbf{b}$ of rational linear inequalities defining $P$ and each inequality in $A\mathbf{x} \le \mathbf{b}$ has size at most $\Lambda$.

Consider an undirected network $G(V,E)$ and the rate-tuple $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$. Let $P$ denote the set of achievable rate-tuples in $\mathbb{R}^N$. Theorem 4 provides a systematic method to characterize the minimal description of $P$ for the general multiple multicast problem. Here we wish to establish upper and lower bounds on the maximum values of the functions that appear in the minimal description of $P$.

The next result strengthens the statement of Theorem 6 and shows that for undirected ring networks we have to consider some distance functions where the maximum edge distance grows at least exponentially in $|E|$. In words, we derive a lower bound on the maximum values of the distance functions needed for the minimum description of the corresponding routing rate region, which is exponential in $|E|$. For that matter, we construct a distance function $g$ that can not be eliminated by any nontrivial distance function with the maximum edge distance less than $2^{\lfloor (|E|-2)/3 \rfloor}$.

**Theorem 7** ((Kakhbod & Yazdi, 2010)). *Let $G(V,E)$ be an undirected ring network with vertices $1,\ 2,\ \cdots,\ |E|$ in a clockwise direction. For $i \in \{1,\ 2,\ \ldots,\ |E|-1\}$, let edge $i$ connect vertices $i$ and $i+1$, and let edge $|E|$ connect vertices $|E|$ and $1$. There exist a distance function $g$ that cannot be eliminated by any nontrivial distance function $f$ with $\max_{e \in E} f(e) < 2^{\lfloor (|E|-2)/3 \rfloor}$.*

*Proof.* Consider a multicast session with $k-1$ destinations, and suppose the source and destination vertices form the set $\{v_1,\ v_2,\ \ldots,\ v_k\}$, where $1 \le v_1 < v_2 < \cdots < v_k \le |E|$. Observe that a minimal spanning subtree is the subgraph consisting of the original network except for the vertices $v_j+1, \ldots, v_{j+1}-1$ and edges $v_j, \ldots, v_{j+1}-1$ for some $j \in \{1, \ldots, k\}$ (with $v_{k+1} = v_1$). Therefore, for any distance function the shortest paths or shortest subtrees for this collection of sessions will correspond to the longest paths $v_j, \ldots, v_{j+1}-1,\ j \in \{1, \ldots, k\}$.

Let $\beta = 2^{\lfloor (|E|-2)/3 \rfloor}$. Suppose we consider the distance function

$$g(e) = \begin{cases} \beta, & e \equiv 1 \pmod 3 \\ 2^{\lfloor (e-2)/3 \rfloor}, & \text{otherwise} \end{cases}$$

and try to find another distance function $f$ that eliminates $g$. Since the shortest broadcast trees are preserved under $f$, it follows that

$$\max_{i \in E} f(i) = f(e),\ e \equiv 1 \pmod 3. \tag{27}$$

Furthermore, for $s \in \{2,\ \ldots,\ \lfloor |E|/3 \rfloor\}$ consider the multicast session consisting of all vertices except $3s-4,\ 3s-3$, and $3s-1$. Under $g$, the path consisting of edges $3s-5,\ 3s-4,\ 3s-3$, and the path consisting of edges $3s-2$ and $3s-1$ are both longest, and therefore (27) implies

$$f(3s-4) + f(3s-3) = f(3s-1),\ s \in \{2,\ \ldots,\ \lfloor |E|/3 \rfloor\}. \tag{28}$$

Finally, for $s \in \{2, \ldots, \lfloor(|E|+1)/3\rfloor\}$ consider the multicast session consisting of all vertices except $3s-4$ and $3s-2$. Under $g$, the path consisting of edges $3s-5$ and $3s-4$ and the path consisting of edges $3s-3$ and $3s-2$ are both longest, and therefore (27) implies

$$f(3s-4) = f(3s-3),\ s \in \{2, \ldots, \lfloor(|E|+1)/3\rfloor\}. \tag{29}$$

By (28) and (29), we see that

$$2f(3s-4) = f(3s-1),\ s \in \{2, \ldots, \lfloor|E|/3\rfloor\}. \tag{30}$$

Equations (27)-(30) imply that $f(e) = f(2)\cdot g(e)$ for all $e \in E$. □

Let $\Lambda^*$ denote the maximum distance among distance functions used for a shortest description of $P$. Theorem 7 establishes that $\Lambda^* \geq 2^{\lfloor(|E|-2)/3\rfloor}$. An extension of 7 to any undirected graph would be straightforward by applying the result of Theorem 7 to the largest cycle of the graph.

**Theorem 8** ((Kakhbod & Yazdi, 2010))**.** *Given undirected graph $G(V,E)$ with maximum cycle length m, for the networking problem in which all possible multicast sessions are supported, the minimum description of the corresponding routing rate region requires a distance function with $\Lambda^* \geq 2^{\lfloor(|E|-2)/3\rfloor}$.*

The next step would be to bound $\Lambda^*$ from above.

**Theorem 9** ((Kakhbod & Yazdi, 2010))**.** *For an undirected or a directed network $G(V,E)$, $\Lambda^* \leq 2^{24|E|^3+8|E|^2}$.*

*Proof.* Theorem 9 is proved by observing the properties of the integer program that define the minimal set of distance functions for describing the routing rate region. Suppose that the distance vector $\mathbf{f} = (f(1),\cdots,f(|E|))$ belongs to the minimal description of $P$. We form the homogeneous set of inequalities $A\mathbf{g} \leq 0$ such that $\mathbf{g} : A\mathbf{g} \leq 0, \mathbf{g} \in \mathbb{Z}^{|E|}$ is the set of all distance vectors that can eliminate $\mathbf{f}$ by the criteria given in Theorem 4. This includes all inequalities that describe the shortest subtrees for every session corresponding to function $f$, and also the non-negativity of elements of $\mathbf{g}$. Notice that this set is non-empty since $\mathbf{f}$ is a solution to it. Furthermore, all elements of matrix $A$ are in $\{0,+1,-1\}$. This implies the upper bound $3|E|+1$ on the size of the inequalities in $A\mathbf{g} \leq 0$. Therefore the facet complexity of $A\mathbf{g} \leq 0$, is at most $\Lambda_A = 3|E|+1$. [(Schrijver, 1998), Theorem 10.2] implies that $A\mathbf{g} \leq 0$ has a rational solution of size at most $4|E|^2\Lambda_A = 12|E|^3+4|E|^2$. Let $\mathbf{g}_r = (p_1/q_1,\cdots,p_{|E|}/q_{|E|})$ denote such a solution. Since $A\mathbf{g} \leq 0$ is a homogeneous set of inequalities, any integral multiple of $\mathbf{g}_r$ is also a solution to $A\mathbf{g} \leq 0$. Now consider the vector $\mathbf{g}_z = (q_1\cdots q_{|E|})\mathbf{g}_r$. Clearly $\mathbf{g}_z \in \{\mathbf{g} : A\mathbf{g} \leq 0, \mathbf{g} \in \mathbb{Z}^{|E|}\}$, so it can eliminate $\mathbf{f}$. Let $\mathbf{g}_z(i)$ be the maximum entry of $\mathbf{g}_z$. Then $size(\mathbf{g}_z(i)) \leq size(q_1\cdots q_{|E|}) + size(\mathbf{g}_r(i))$. Since $size(q_1\cdots q_{|E|}) \leq size(\mathbf{g}_r)$ and $size(\mathbf{g}_r(i)) \leq size(\mathbf{g}_r)$, then $size(\mathbf{g}_z(i)) \leq 24|E|^3+8|E|^2$. This yields the result. □

The following probabilistic result, suggests that a small fraction of the distance functions in our characterization of the fractional routing capacity region are truly needed and that most distance functions can be eliminated by distance functions where the maximum entry grows polynomially with $|E|$.

**Theorem 10** ((Kakhbod & Yazdi, 2010))**.** *Let $G(V,E)$ be an undirected ring network with edges labeled $1, 2, \cdots, |E|$ in a clockwise order. Choose any integer $m \geq 6$, and suppose $g_{\max} = \max_{e\in E} g(e) > g^* \doteq |E|^m/(1-\frac{|E|^m}{g_{\max}})$. Assume without loss of generality that $g(|E|) = g_{\max}$ and for $e \in E \setminus |E|$ let $g(e)$ be uniformly chosen among nonnegative integers less than or equal to $g_{\max}$. Then with probability at least $1-\frac{4}{|E|^{m-5}}-\frac{1}{|E|^{m(|E|-1)}}$ we can find a distance function $f$ with $f_{\max} \leq g^*$ that eliminates $g$.*

*Proof.* Given distance function $g$ with $g_{\max} \geq g^*$, let $\phi = \lfloor g_{\max}/|E|^m \rfloor$, and define

$$f(e) = g(e) - (g(e) \pmod{\phi}),\ e \in E.$$

Distance function $f$ eliminates distance function $g$ if for any pair of edge-disjoint subtrees $E_1$ and $E_2$ of $E$, the condition $\sum_{e\in E_1} g(e) \leq \sum_{i\in E_2} g(e)$ implies $\sum_{e\in E_1} f(e) \leq \sum_{i\in E_2} f(e)$. Let $\mathcal{E}_{E_1,E_2}$ be the event that $\sum_{e\in E_1} g(e) \leq \sum_{e\in E_2} g(e)$ and $\sum_{e\in E_1} f(e) > \sum_{e\in E_2} f(e)$. Define

$$\begin{aligned} \Delta_g &= \sum_{e\in E_1} g(e) - \sum_{e\in E_2} g(e) \\ \text{and } \Delta_f &= \sum_{e\in E_1} f(e) - \sum_{i\in E_2} f(e). \end{aligned}$$

Since $0 \leq g(e) - f(e) < \phi$ for all $e \in E$, it follows that

$$|\Delta_g - \Delta_f| \ \leq\ \sum_{e\in E} |g(e) - f(e)| \ <\ \phi \cdot |E|.$$

We know that $\Delta_g \leq 0$ and $\Delta_f > 0$, and so $|\Delta_g| < \phi \cdot |E|$. Let $E_{\min} = \min_{e\in E_1\cup E_2} e$. Observe that $E_{\min} \neq |E|$. Given $g(e), \leq e \in E \setminus E_{\min}$, there are at most $2\phi \cdot |E|$ choices for $g(E_{\min})$ that result in $-\phi \cdot |E| < \Delta_g \leq 0$. Furthermore $g(E_{\min})$ is a random variable uniformly distributed over the integers between 0 and $g_{\max}$. Therefore,

$$\mathbb{P}(\mathcal{E}_{E_1,E_2}) \ \leq\ \frac{2\phi \cdot |E|}{g_{\max}+1} \ <\ \frac{2 \cdot \frac{g_{\max}}{|E|^m} \cdot |E|}{g_{\max}} \ =\ \frac{2}{|E|^{m-1}}.$$

The number of pairs of edge-disjoint subtrees $E_1$ and $E_2$ we need consider is less than $2|E|^4$. Hence,

$$\mathbb{P}\left(\bigcup_{E_1,E_2} \mathcal{E}_{E_1,E_2}\right) \ <\ 2|E|^4 \cdot \frac{2}{|E|^{m-1}} \ =\ \frac{4}{|E|^{m-5}}.$$

In addition, In order to omit the trivial cases, a distance function is not trivial if and only if there exist at least $e$ and $e'$ in $E$ such that $f(e) > 0$ and $f(e') > 0$ , since for all $e \in E$, $\mathbb{P}(f(e) = 0) < \frac{1}{|E|^m}$, Thus, with probability at least $1 - \frac{4}{|E|^{m-5}} - \frac{1}{|E|^{m(|E|-1)}}$ we can use distance function $f$ to eliminate $g$. Since $f(e) \pmod{\phi} = 0$ for all $e \in E$, we can eliminate distance function $f$ by distance function $f^*$ with $f^*(e) = f(e)/\phi,\ e \in E$. Notice that for all $e \in E$,

$$\begin{aligned} f^*(e) \ &=\ \tfrac{f(e)}{\phi} \ \leq\ \tfrac{g(e)}{\phi} \ \leq\ \tfrac{g_{\max}}{\lfloor g_{\max}/|E|^m \rfloor} \ <\ \tfrac{g_{\max}}{\frac{g_{\max}}{|E|^m}-1} \\ &=\ \tfrac{|E|^m}{1-\frac{|E|^m}{g_{\max}}}. \end{aligned} \tag{31}$$

□

## 4 Conclusion

In this article we studied a systematic approach to characterizing the set of achievable rates by routing in an arbitrary network. We introduced an elimination method that reduced the infinite description of routing rate region into a minimal and finite set. The elimination technique leads to a simple characterization of the routing capacity of ring networks in two scenarios of multiple multicast sessions. Furthermore, we studied the computational complexity of the routing rate region in terms of the maximum value of distance functions in the minimal necessary and sufficient set of distance function.

There are several directions for further extension of the results in this article. While routing is traditionally defined for wired networks, one can study routing in wireless networks. The recent linear deterministic model of wireless networks (Avestimehr et al., 2011) provides a simple yet powerful framework for the study of routing schemes in wireless networks. In this direction, one can study the generalization of the Japanese theorem to wireless networks and the minimal set of inequalities that describe the routing rate region.

In this article we focused on the exact characterization of the routing rate region. We observed that the complexity of the exact characterization increases exponentially with the size of the network. As a more practical approach, one can study the effect of limiting the maximum value of distance functions into some small number. Given the approximate max-flow min-cut result of Leighton and Rao (Leighton & Rao, 1999) we expect that even distance functions of zero and one values provide tight approximation of the routing rate region in a broad class of networks.

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