Heterogeneous Learning in Product Markets*

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Abstract

We examine the impact of market structures and asymmetries in learning technologies on trade within a product market, where a new product of unknown quality is introduced to compete with an existing product of known quality. Our findings indicate that market efficiency is attained under both monopoly and competition when buyers are symmetric in how much information they generate when they consume the new product. However, when buyers exhibit asymmetry in information generation, only the monopolistic market achieves efficiency. Inefficiencies arise due to the information externality buyers with inferior learning technology generate.

Keywords: Market structure, Market power, Trading, Learning.

JEL Classification: C7

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1 Introduction

E-commerce marketplaces offer a convenient platform for sellers to engage in competition and introduce new products to the market. These online platforms facilitate the selling process and serve as a space where consumers can share their opinions and experiences with products through reviews. These consumer reviews play a crucial role in providing valuable insights into the quality and potential for adopting new products and have implications for the pricing of old (existing) products.[1]

In these markets, the abundance of individual reviews implies that the perception of a new product is approximately the same across market participants. However, an essential aspect of *heterogeneity* among agents still exists: the feedback and reviews vary in format (ranging from simple 1-to-5-star ratings to detailed written comments) and accuracy levels. This diversity in feedback is a natural outcome, as buyers have different experiences with the product. Even if all consumers are focused on using the best product, some may adapt more quickly and utilize additional features more often, producing more information about the product's quality. The fact that these varied experiences contribute to a collective public belief is highly reflective of today's markets, characterized by frequent and comprehensive feedback on consumer experiences through surveys or published reviews.

Considering this heterogeneity, how do reviews influence price competition and the learning process of Bayesian market participants? At what point does a buyer choose to buy a new product? Does competition improve overall welfare? What kinds of distortions emerge due to competition? Furthermore, what policies can be implemented to boost market efficiency? In this paper, we address these questions, with particular emphasis on the impact of buyer heterogeneity in the context of providing reviews.

To investigate these questions, we examine a market for indivisible experience goods, where an unknown quality new product is introduced to compete with an existing (old) product with known quality. The new product has two possible quality levels, with

[1] User reviews, particularly on sites like Amazon, mean a great deal to shoppers. "*A product that has just one review is 65% more likely to be purchased than a product that has none*", according to Matt Moog, CEO of Power Reviews. He added that one-third of online shoppers refuse to purchase products without positive customer feedback.

the true state unknown. Market participants are Bayesian agents who can progressively learn the state through reviews from buyers who have experienced the new product. Crucially, we allow for buyer heterogeneity in the (expected) information they generate about the new product's quality by using it. This represents a reduced form version of their feedback's varying (exogenous) accuracy. As their product usage leads to greater learning about product quality, we label agents providing more accurate signals as better learners.[2]

To analyze the impact of market power, we examine the efficient consumption patterns and the decentralized market outcomes (i.e., the Markov perfect equilibria) under both a monopolistic firm offering both products and competition between two specialized firms, each providing one of the products. In this context, efficiency is defined as the consumption patterns that maximize the expected total surplus for all market participants.

We first establish that both efficient and decentralized market outcomes exhibit a series of belief thresholds. In the early stages, when confidence in the new product's quality is low, only the best learners consume it. As time progresses, if the best learners' reviews significantly enhance the market's confidence in the new product, worse (less proficient) learners also begin to purchase it. In other words, all solutions include a **beta phase** during which only the best learners experience the new (unknown) product. We explicitly calculate the beta phase and its expected duration based on the endogenous model parameters.

We then explore the efficiency properties of various market structures, focusing on comparing monopoly and competition. The primary finding is that the welfare performance of monopolistic and duopolistic market structures heavily depends on the learning technology among buyers. We demonstrate that both market structures result in efficiency when buyers have homogeneous learning technologies. However, when buyers are heterogeneous, competitive markets lose efficiency while monopolistic markets preserve it.

[2]The assumption of exogenous learning ability is approximately correct in many instances of online commerce, where the differences in the amount of feedback mostly relate to individual attitudes instead of strategic considerations. Consequently, sellers can enhance information production only by targeting the best learners. We think having sellers who endogenously increase the agent's feedback may be relevant for some applications. However, we do not follow this route in this paper because we are interested in isolating the informational externalities due to learning asymmetries.

The lower welfare resulting from competition might appear counterintuitive. The underlying reasoning behind these findings is that, in dynamic (oligopolistic or monopolistic) markets, bilateral contracting between two parties generates learning externalities on other market actors, proportional to their value of information. However, in a monopoly, the monopolist's optimal pricing nullifies the value of information for all buyers, even under asymmetric learning technologies.[3]

Instead, under competition, a portion of these externalities benefits potential buyers not involved in the transaction, and thus, it is not internalized in prices. This situation resembles a new retailer introducing a product on Amazon and offering discounts to a subset of consumers capable of providing detailed reviews. Intuitively, the discount amount increases with the entrant's future market power in the event of success. Our findings emphasize that competition may lead to welfare-reducing discounting strategies.

Our model also offers new perspectives on the form of inefficiency caused by competition. Notably, just like in the optimal (the first-best) scenario, the equilibrium exhibits a threshold structure. Only the best learners adopt the new product at a low confidence level, while the worse learners move to the new product as the public confidence level grows. Importantly, this equilibrium features efficiency for the top learners, as the belief threshold for engaging the best learners is the same as the first-best scenario. Therefore, all the new products that are sufficiently promising (i.e., their prior market belief is high enough) are given a chance. However, competition distorts the threshold required to exit the beta phase and begin serving the entire market.

We also delve deeper into the comparative statics of the inefficiency above, specifically demonstrating that although asymmetries in learning technology are necessary for an inefficient market outcome, the magnitude of the distortion is *not* monotone in the amount of heterogeneity.

Lastly, we explore a potential solution to the distortions caused by competition. We demonstrate that implementing multilateral contracts results in an efficient equilibrium outcome. Specifically, we enhance the commitment power of the sellers by allowing them to make take-it-or-leave-it offers to multiple market participants. Put succinctly, these

[3]Importantly, this conclusion does not depend on the monopolist's awareness of the consumer type.

take-it-or-leave-it offers to multiple market participants compensate good learners for allowing the product to be consumed by less proficient learners, offsetting the information externality costs generated by the latter. We establish that if such contracts are feasible, the decentralized outcome achieves efficiency, irrespective of the heterogeneity in learning technologies.

1.1 Related Literature

This paper contributes to the growing literature on pricing with externalities, learning, strategic pricing, and experimentation with technological innovation. The recent literature has a variety of focuses. For example, big data and learning from reviews (e.g., Acemoglu et al. (2022a), Acemoglu et al. (2022b)), design of crowdfunding campaigns (e.g., Alaei, Malekian and Mostagir (2016)), optimal design of contests (e.g., Bimpikis, Ehsani and Mostagir (2019)), information diffusion in networks (e.g., Candogan and Drakopoulos (2020)), the different implications of vertical and horizontal differentiation of firms (e.g., Koh and Li (2023)) opinion dynamics (e.g., Jadbabaie et al. (2012); Cerreia-Vioglio, Corrao and Lanzani (2023)), strategic information exchange (e.g., Candogan and Strack (2023)), pricing with local externalities (e.g., Candogan, Bimpikis and Ozdaglar (2012)), information sharing and online platforms (e.g., Li and Hitt (2008), Che and Hörner (2018), Bonatti and Cisternas (2020),Vellodi (2021)), the effect of different ambiguity attitudes on learning (e.g., Battigalli et al. (2019)).[4]

Most relevantly, Acemoglu et al. (2022b), like us, single out an externality induced by some consumers on others. However, they argue there is partial overlapping in the private information of the different consumers, and the information provided by one consumer depresses the value of the information of the others. In contrast to this body of literature, our analysis highlights the importance of competition dynamics among sellers in cases where agents generate externalities while displaying heterogeneity in their learning quality. Aleksenko and Kohlhepp (2023) also looks at the impact on dynamic pricing of reviews that can be heterogeneous in terms of quality. Differently from us, they study a Poisson information structure, short-lived consumers, monopoly rather than oligopoly,

[4]See also Bergemann and Bonatti (2019) for an excellent survey of this literature.

and, more importantly, their heterogeneity in the propensity to review is realized after the purchase decision has been made, not living scope for price differentiation from the sellers. At the same time, their model allows for private information, while we consider a public information environment.

More broadly, our paper is related to the literature on dynamic pricing.[5] In general, time-varying prices may arise for a variety of reasons. For example, they might be due to learning about new experience goods (e.g., Caminal and Vives (1999)), uncertainty about the opponents' production costs (Bonatti, Cisternas and Toikka (2017)), data sharing (e.g., Acemoglu et al. (2022b)), product choice with social learning (e.g., Maglaras, Scarsini and Vaccari (2020)), scarcity of the products with regard to the number of buyers (e.g., Gallego and van Ryzin (1994)), network externalities (e.g., Cabral, Salant and Woroch (1999)), stochastic incoming demand (e.g., Board (2008)), forward-looking buyers who enter the market over time (e.g., Board and Skrzypacz (2016)), and time-varying values of buyers (e.g., Stokey (1979), Stokey (1981)).[6] By contrast, we consider dynamic pricing when consumers differ in the precision of the information they generate on product quality. This is crucial and leads to rich predictions about how the sellers discount and price discriminate between consumers based on their learning technologies. It also allows us to explore different questions, like the relative efficiency performance of monopoly and competition and how the inefficiency depends on the heterogeneity of the buyers. Several papers in this literature highlight that competition can harm welfare (e.g., Bergemann and Välimäki (1997, 2000), Fang, Noe and Strack (2020)). Still, they do not study the impact of heterogeneity in the learning technology of the agents.

Finally, our paper is linked with works that study big data and the use-based evolution of beliefs about the quality of a product. Related questions to this type of belief dynamics have been addressed in different frameworks in several papers (e.g., Bolton and Harris (1999), Décamps, Mariotti and Villeneuve (2006), Acemoglu, Bimpikis and Ozdaglar (2011), Begenau, Farboodi and Veldkamp (2018), Cetemen, Urgun and Yariv (2023)).[7] In contrast to these works, we consider how the availability of information

[5] Phillips (2005) extensively reviews this topic.

[6] Another cause suggested in the literature for varying prices over time is information diffusion due to the word-of-mouth effect (e.g., Ajorlou, Jadbabaie and Kakhbod (2018)).

[7] For example, Park (2001) observed a possible linkage between learning asymmetries and efficiency.

through heterogeneous quality sources affects welfare, trading volume, the beta phase, and interacts with market power (monopoly and competition). We further present policies that can reduce distortions.

The rest of the paper proceeds as follows. Section 2 introduces our formal model, and Section 3 studies the first-best consumption allocation. Section 4 moves to the analysis of the decentralized outcome and presents our main results. Section 5 proposes multilateral contracts for sellers and studies the characteristics of the beta phase. Section 6 concludes.

2 Model

In a market with two distinct products, consumers encounter an established product a with a known payoff and a recently introduced product b with an expected payoff unknown to sellers and buyers. There are $M \in \{1, 2\}$ sellers and 2 possibly asymmetric buyers.[8] When $M = 1$ a profit-maximizing **monopolist** sells both products. In the **duopoly** structure where $M = 2$, two different sellers compete strategically to sell the products; one seller sells the new product, and the other sells the established one. In this case, we will label each seller as the product he sells.

2.1 Buyers' asymmetry and flow payoffs

We assume each product generates the following payoff flow for its buyer(s). The old (established) product a generates a (cumulative) consumption utility C_{ai} for buyer $i \in \{1, 2\}$ following the dynamics

$$dC_{ai}(t) = \mu_a dt,$$

when the buyer i consumes the old product. If buyer i consumes the new product b they receive a (cumulative) consumption utility C_{bi} following the dynamics

$$dC_{bi}(t) = \theta dt + \sigma_i dZ_{it},$$

Still, he studies neither when it is possible nor the form of the inefficiency.

[8] We focus on the case of two types of buyers as it already conveys the key intuitions behind our analysis. All the results hold in the case of an arbitrary (but finite) number of heterogeneous buyers

where Z_{it}, $i \in \{1, 2\}$, are independent standard Brownian motions (BMs). Therefore, when consumed, the expected experienced consumption utility of the new product grows at the rate θ, whose value is unknown to sellers and buyers. We assume that $\theta \in \{\ell, h\}$ with $\ell < h$. Both sellers and buyers know these two alternative values but do not know the true value of θ. We assume that the problem is not trivial; that is, $\ell < \mu_a < h$. Therefore, the value of θ determines the objectively preferable product, which is the same for every buyer. All the market participants share a common prior $\Pr\{\theta = h\} = \pi_0$ at time 0 when the new product becomes available, they update the posterior distribution of θ as new information arrives.

The volatility σ_i indicates how noisy the experience of buyer i reflects the product's actual consumption utility. Importantly, we let $\sigma_1 \geq \sigma_2$. This means buyers may have varying levels of accuracy in evaluating the new product, resulting in **asymmetry**. A buyer with a lower σ value is considered a better learner. Clearly, when $\sigma_i = \sigma$ for all $i \in \{1, 2\}$, then buyers are **symmetric**. To isolate the effect of heterogeneous learning technology (i.e., heterogeneous σ_i), we assume that the buyers are otherwise identical and, in particular, that they share the same valuation for the product of unknown quality in both situations, i.e., h and ℓ are the same across buyers.[9]

2.2 Trading volume and payoffs

At any instant, a buyer can experience at most one product. Thus, at time t, buyer i's consumption falls into one of three categories: $\{a, b, \emptyset\}$, with $\emptyset$ indicating that buyer i does not consume any product. We denote by $\{\xi_{ik}(t)\}_{t \in \mathbb{R}_+}$ the (measurable) allocation process, whose value at time t is either 0 or 1 such that $\xi_{ik}(t) = 1$ if buyer i consumes product $k \in \{a, b\}$ at time t and $\xi_{ik}(t) = 0$ otherwise.

Both buyers and sellers are risk-neutral and forward-looking. They discount payoffs exponentially at a shared rate $\rho > 0$. Sellers have all the bargaining power; that is, offers are in take-it-or-leave-it forms. At time t, the price of product k for buyer i posted by its seller is $p_{k,i}(t)$.[10]

[9] Our main results continue to hold as long as the ordinal valuations of the different buyers are the same.

[10] Seller's price $p_{k,i}$, $k \in \{a, b\}$ and $i \in \{1, 2\}$, is admissible if it is a process progressively measurable with

For a given strategy and price pair $(\xi, p) = (\xi_{ik}(t), p_{k,i}(t))_{i\in\{1,2\},k\in\{a,b\},t\in\mathbb{R}_+}$, the payoff of buyer i is given by

$$U_i^B(p, \xi, \pi) = \mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} \sum_{k\in\{a,b\}} \underbrace{\xi_{ik}(t)}_{\text{order}} \Big(\underbrace{dC_{ki}(t)}_{\text{flow gain}} - \underbrace{p_{k,i}(t)}_{\text{payment}}\, dt\Big)\Big]. \tag{1}$$

Without loss of generality, we normalize the production cost to 0 so that the payoff of the sellers equals the total revenues they obtain from the products they sell. Importantly, how we compute these revenues depends on the market structure (monopoly versus duopoly). Below, we present the expected discounted payoffs in the two cases considered in the paper.

Monopoly. When there is a unique seller of both products, the seller's payoff is given by

$$U_m(p, \xi, \pi)) = \mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} \underbrace{\Big(\sum_{i=1}^{2} \xi_{ia}(t) p_{a,i}(t) + \sum_{i=1}^{2} \xi_{ib}(t) p_{b,i}(t)\Big)}_{\text{overall time } t \text{ monopoly profit (sale)}} dt\Big]. \tag{2}$$

Under competition, the objective functions of the two sellers are similar, but each considers only the profits generated from their respective sales.

Duopoly. Under duopoly, the payoff of seller $k \in \{a, b\}$ is given by[11]

$$U_k^S(p, \xi, \pi) = \mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} \underbrace{\sum_{i=1}^{2} \xi_{ik}(t) p_{k,i}(t)}_{\substack{\text{time } t \text{ seller } k \\ \text{profit (sale)}}} dt\Big]. \tag{3}$$

respect to public information and satisfies the integrability condition $\mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} p_{k,i}(t)\, dt\Big] < \infty$, where $\mathbf{E}[\cdot]$ denotes the expectation.

[11]Due to the integrability of $p_{k,i}$ and the boundedness of θ and $\xi_{ik}(t)$, all payoffs in (1), (2), and (3) are finite.

2.3 Belief dynamics

At each time t, which product the buyers experience and their flow of consumption utilities are public information. Therefore, even though the amount of information buyers produce differs, there is a unique market belief about the type of the unknown product. Formally, let $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by the public information available up to time t, that is, the filtration generated by the public signal $(\xi(t), \mathbf{X}(t)))_{t\geq 0}$, where $\mathbf{X}(t) = (X_1(t), X_2(t))$ is defined by

$$X_i(t) = \int_0^t \xi_{ib}(\tau) dC_{bi}(\tau), \quad \forall i = 1, 2.$$[12]

With this information structure, the public belief is denoted as

$$\pi_t := \Pr\left\{\theta = h | \mathcal{F}_t\right\}.$$

The following lemma characterizes the dynamics of the market belief in terms of the (endogenous) trading volume and learning technologies (i.e., σ_i) of buyers.[13]

Lemma 1. *[Belief Evolution] We have*

$$d\pi_t = \pi_t(1-\pi_t)(h-\ell)\sqrt{\sum_{i=1}^{2} \frac{\xi_{ib}(t)}{\sigma_i^2}} dZ_t$$

where Z_t is a standard Wiener process with respect to the filtration $\mathcal{F}_t$. In particular, in the case of symmetric buyers, we have

$$d\pi_t = \frac{\pi_t(1-\pi_t)(h-\ell)}{\sigma}\sqrt{Vol_b(t)} dZ_t,$$

where $Vol_b(t) = \sum_{i=1}^{2} \xi_{ib}(t)$.

[12] Public information depends on agents' behavior in equilibrium. Therefore, the public information structure is part of our equilibrium definition.

[13] The proof of this lemma resembles the proof of Lemma 1 of Bolton and Harris (1999), with some minor differences. Here, the heterogeneous learning technologies play the same role as the intensity of experimentation in that paper.

Next, we leverage the previous result on the public belief dynamics to study the optimal choice of buyers and the dynamic pricing of sellers under different market structures. First, we study the optimal consumption pattern for a planner who wants to maximize the sum of the utilities of the market participants. Then, we consider the decentralized equilibrium that arises when each market participant best replies to the opponents' strategy, and we explore the difference between these two situations.

3 The first-best— efficient strategies

The first-best formulation. In this section, we consider the social welfare-maximizing strategies; that is, we specify strategies that maximize the sum of the utilities of all market participants.

Given buyers' and sellers' payoffs (see (1)-(3)), the payments cancel each other out in the welfare-maximization problem. As a result, the objective function is the discounted sum of the consumption utility of the buyers:[14]

$$W(\pi) = \max_{\xi_{ik}} \mathbf{E}\left[\sum_{i=1}^{2} \sum_{k\in\{a,b\}} \int_0^\infty \rho e^{-\rho t} \xi_{ik}(t) dC_{ki}(t)\right].$$

Therefore, efficiency only depends on the consumption of each agent, regardless of the transfers. Given that the system is time-invariant, the optimal ξ_{ik} only depends on the public belief π, and the maximization can be mapped into an optimal stopping problem (e.g., Karatzas (1984)). With this, the Hamilton-Jacobi-Bellman (HJB) equation for this problem is given by:

$$W(\pi) = \max_{(\xi_{ik}\in\{0,1\}:\xi_{ia}+\xi_{ib}\leq 1)_{i\in\{1,2\},k\in\{a,b\}}} \left\{ \sum_{i=1}^{2} (\xi_{ia}\mu_a + \xi_{ib}\mathbf{E}_\pi[\theta]) + W''(\pi) \sum_{i=1}^{2} \xi_{ib} \frac{g\big(\pi, h, \ell\big)}{2\rho\sigma_i^2} \right\}$$

where $g(\pi, h, \ell) = \big((h-\ell)\pi(1-\pi)\big)^2$. Since the planner's instantaneous gain from allocating

[14] To be feasible, the maximization is performed over strategies that only reflect the available information. Formally, the consumption allocation process $\{\xi_{ik}(t)\}_{t\in\mathbb{R}_+}$ for buyer $i \in \{1,2\}$ and product $k \in \{1,2\}$ takes values in $\{0,1\}$ and is progressively measurable with respect to the filtration $\mathcal{F}$.

consumers to the risky good is linear in π_t, the efficient allocation is pinned down by a simple sequence of cutoffs on the public belief $(\pi_{\text{fb},i})_{i=1}^{2}$. Consumer i buys product b at time t if and only if $\pi_t > \pi_{\text{fb,i}}$, i.e.,

$$\xi_{ib}(t) = \mathbb{I}_{(\pi_{\text{fb,i}},\infty)}(\pi_t) \qquad \forall\, i \in \{1, 2\}. \tag{4}$$

3.1 Symmetric buyers

In the case of symmetric buyers, there is only one cutoff π_{fb}, i.e., $\pi_{\text{fb},1} = \pi_{\text{fb},2} = \pi_{\text{fb}}$. In this case, the first-best cutoff and welfare have closed-form expressions summarized by the next two propositions.

Proposition 1. *The first-best (social welfare) maximizing cutoff π_{fb} is increasing in μ_a, σ^2, and ρ. It is decreasing in h. The first-best social welfare $W(\pi)$ equals $2\mu_a$ for $\pi \leq \pi_{\text{fb}}$ and is strictly convex in π when $\pi_{\text{fb}} < \pi \leq 1$.*

Since the known product acts as an outside option, a higher μ_a is easily seen to induce a higher π_{fb}. On the other hand, a larger h increases the value of choosing alternative b through a higher instantaneous value given a particular belief and a higher learning value $(h - \ell)$. Therefore, it unambiguously induces a lower π_{fb}. The effects of the information processing technology and the discount factor are intuitive.

Intuitively, the welfare function is flat before the optimal cutoff. The per-consumer value there equals the flow of payoff guaranteed by the known product bought by all the consumers. After the cutoff, the value increases with the probability assigned to the high quality of the unknown product. There, the convexity of the welfare is due to the known product acting as an outside option. Next, we move to the more interesting case of different learning technologies for the buyers.

3.2 Asymmetric buyers

Recall that we have $\sigma_1 > \sigma_2$. It is useful to consider the average (per consumer) first-best welfare. It satisfies the following HJB:

$$W_{avg}(\pi) = \max_{\xi_{1b},\xi_{2b}\in\{0,1\}} \left\{ \mu_a + \sum_{i=1}^{2} \frac{\xi_{ib}}{2} \left(\mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)W''_{avg}(\pi)}{\rho\sigma_i^2} \right) \right\}. \tag{5}$$

Denote ξ_{1b}^*, ξ_{2b}^* the maximizer of the average utility above. The optimality condition in each ξ_{ib} yields

$$\xi_{ib}^* = \begin{cases} 1, & \text{if } \mathbf{E}_\pi[\theta] + \frac{g(\pi,h,\ell)W''_{avg}(\pi)}{\rho\sigma_i^2} \geq \mu_a; \\ 0, & \text{otherwise.} \end{cases}$$

When W_{avg} is convex (which is verified after (5) is solved numerically), $\sigma_1 > \sigma_2$ and the previous expression of ξ_{ib}^* imply that

$$\xi_{1b}^* = 1 \quad \Longrightarrow \quad \xi_{2b}^* = 1.$$

Therefore, we conjecture that the optimal strategy is a threshold type.[15] There exist thresholds $\pi_{\text{fb},1}, \pi_{\text{fb},2} \in (0,1)$ with $\pi_{\text{fb},1} > \pi_{\text{fb},2}$, such that

$$\xi_{ib}^*(t) = 1_{\{\pi_t > \pi_{\text{fb},i}\}}, \quad \forall i \in \{1,2\}. \tag{6}$$

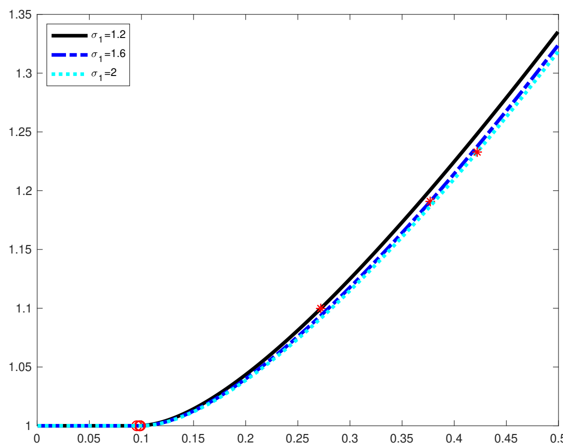
The structure of this optimal policy is simple. There are two thresholds $\pi_{\text{fb},1}$ and $\pi_{\text{fb},2}$ with $0 < \pi_{\text{fb},2} < \pi_{\text{fb},1} < 1$. Both buyers purchase the new product b when the public belief is higher than $\pi_{\text{fb},1}$, no buyer purchases b when the public belief is lower than $\pi_{\text{fb},2}$, and only the buyer with the better learning technology purchases the new product when $\pi_{\text{fb},2} < \pi < \pi_{\text{fb},1}$. The intuition for the lower threshold for the better learner is simple. First, note that information is valuable for overall welfare because it allows better consumption choices for the consumers (this mathematically translates into the convexity of the value function.) Second, the higher signal precision of the better learner implies that he can trade-off exploitation in favor of information generation at more favorable terms, and

[15] Proposition 8 in the Appendix verifies this conjecture.

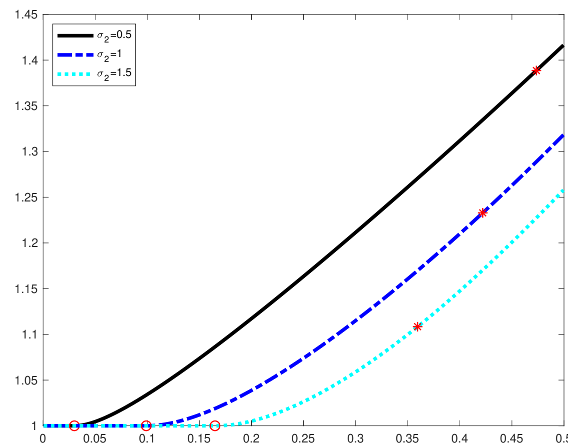
therefore, it is optimal to start to do so for more pessimistic public beliefs.

The following proposition summarizes our results for the first best problem with asymmetric buyers. Denote as π_{myopic} the belief such that $\mathbf{E}_{\pi_{\text{myopic}}}[\theta] = \mu_a$.

Proposition 2. *The first-best policy is in cutoff strategies with* $\pi_{\text{myopic}} > \pi_{\text{fb},1} > \pi_{\text{fb},2}$.



(a) Different σ_1



(b) Different σ_2

Figure 1: Average value and cutoffs in the first best problem with asymmetric buyers. In both panels, the horizontal coordinate of the red circle is $\pi_{\text{fb},2}$, and the horizontal coordinate of the red asterisk is $\pi_{\text{fb},1}$. In the left panel, the average value decreases, and $\pi_{\text{fb},1}$ increases with σ_1. In the right panel, the average value decreases, $\pi_{\text{fb},2}$ increases, and $\pi_{\text{fb},1}$ decreases with σ_2. Other paramaters: $h = 2$, $\ell = 0$, $\mu_a = 1$, $\rho = 0.5$, $\sigma_2 = 1$ (left panel), and $\sigma_1 = 2$ (right panel).

Figure 1 illustrates how the average values and cutoffs depend on σ_1 and σ_2. In the next sections, we compare these first-best cutoffs with the ones obtained under the two different competition structures presented above: a monopolist selling both products and competition between two sellers.

4 Analysis: Decentralized Outcome

We aim to find equilibrium strategies when buyers are symmetric and asymmetric in their learning technology σ_i. Specifically, our goal is to uncover the joint influence of the market structure and the asymmetries in the learning technologies on learning processes, trading volume, and overall efficiency.

In what follows, we will show that, if the buyers have asymmetric learning technologies, a monopolistic market structure is efficient, while competition induces a welfare loss. Notably, a monopoly is efficient both when the monopolist can observe the learning technologies of the buyers and when the monopolist cannot. Formally, to deal with both cases, we use a notation that allows the seller to discriminate between buyers, depending on their learning technology. Therefore the choice variable for the monopolist is the vector $(p_{a,i}, p_{b,i})_{i\in\{1,2\}}$. Similarly, under competition, the choice variable for seller a is $(p_{a,i})_{i\in\{1,2\}}$ and for seller b is $(p_{b,i})_{i\in\{1,2\}}$.

We note that the negative result we will obtain for competition is only reinforced by the assumption that the sellers can price discriminate on the basis of the quality of information produced. Indeed, it is well known (see, e.g., the textbook treatment of Wilson (1993)) that, even in static markets, the combination of market power for the seller, asymmetric consumers, and the impossibility of discriminating between consumers creates inefficiencies. However, inefficiencies are usually avoided when the seller can discriminate. Contrarily, we will show that, in dynamic markets with learning, discrimination is insufficient to eliminate the inefficiencies caused by competition.

To analyze this model, we restrict our attention to Markov perfect equilibria. Let $\Pi = \times_{k\in\{a,b\}} \times_{i\in\{1,2\}} \Pi_{ki}$ be the set of almost everywhere bounded functions mapping public histories into $\mathbb{R}^4$. They correspond to the possible pricing strategy of the sellers. Given the timing of offers, the Markov restriction implies that the pricing strategy $p_{k,i} \in \Pi_{ki}$ for each product $k \in \{a, b\}$ can be written as a function mapping the current public belief to the real numbers, and the strategy of buyer i, purchasing choices ξ_{ik}, $k \in \{1, 2\}$, can be written as a function mapping the current public belief and the posted prices to $\{0, 1\}$. We denote this class of functions by Ξ_{ik}.

We state the relevant equilibrium notion for the case of competition. An analogous definition that takes into account that the choice variable of the monopolist has two dimensions is used for the study of the monopoly case.

Definition 1. *A collection of Markov strategies (ξ^*, p^*) is a Markov Perfect Equilibrium if for all*

$k \in \{a, b\}, i \in \{1, 2\}, (p_{k,1}, p_{k,2}) \in \Pi_{k1} \times \Pi_{k2}, (\xi_{ia}, \xi_{ib}) \in \Xi_{ia} \times \Xi_{ib}$, *and* $\pi \in (0, 1)$, *we have*

$$U_k^S(p^*, \xi^*, \pi) \geq U_k^S\left(p_k, p^*_{-k}, \xi^*, \pi\right) \qquad \textit{and} \qquad U_i^B(p^*, \xi^*, \pi) \geq U_i^B\left(p^*, \xi^*_{-i}, \xi_i, \pi\right).$$[16]

4.1 Monopoly

We start by proving that the revenue-maximizing policy of a monopolist is *efficient*; that is, the induced consumption pattern maximizes the total surplus, independent of the learning technologies. To prove this result, we first derive the buyers' and the monopolist's HJB equations. Recall that we assume that the sellers and, in this particular case, make offers in take-it-or-leave-it form. Therefore, the HJB equation of a buyer i captures the comparison between the two products at the posted prices:[17]

$$v_i(\pi) = \max\left\{ \underbrace{\mu_a - p_{a,i}(\pi)}_{\text{flow gain from } a}, \underbrace{\mathbf{E}_\pi[\theta] - p_{b,i}(\pi)}_{\text{flow gain from } b} + \underbrace{\frac{g(\pi, h, \ell)}{2\rho\sigma_i^2} v_i''(\pi)}_{\text{learning gain from } i \text{ buying } b}, 0 \right\}$$
$$+ \underbrace{g(\pi, h, \ell) \sum_{j \neq i} \frac{\xi_{jb}\left(\pi, p_a(\pi), p_b(\pi)\right)}{2\rho\sigma_j^2} v_i''(\pi)}_{\text{learning gain from others buying } b}. \tag{7}$$

Each term of the above HJB equation has two parts. If buyer i buys the product a of known quality, then $\mu_a - p_{a,i}(\pi)$ is the instant (expected) flow payoff, and

$$g(\pi, h, \ell) \sum_{j \neq i} \frac{\xi_{jb}\left(\pi, p_a(\pi), p_b(\pi)\right)}{2\rho\sigma_j^2} v_i''(\pi)$$

[16] Because the pricing functions are almost everywhere bounded, $p_{ik}(t) = p_{ik}(\pi_t)$ satisfies the integrability condition $\mathbf{E}\left[\int_0^\infty \rho e^{-\rho t} p_{ik}(t, \xi(t)) dt\right] < \infty$. Therefore, the values for buyers and sellers are all finite in Definition 1. We will show later that the equilibrium prices depend on the second-order derivatives of buyers' and sellers' values functions, which are bounded. Therefore, the equilibrium prices are almost everywhere bounded as well.

[17] The dynamic programming principle implies that the value function defined in (1) is a viscosity solution to (7), see, e.g., (Pham, 2009, Proposition 4.3.1 and 4.3.2) and the last section of the Online Appendix.

is the expected continuation payoff (which is due to learning). A similar decomposition holds when buyer i buys the risky product b of unknown quality, but then the expected amount of information generated is increased by a term inversely proportional to σ_i^2, and the (expected) flow payoff becomes $\mathbf{E}_\pi[\theta] - p_b(\pi)$. Buyer i's strategy $\xi_{ik}(t)$ is determined by which term in the first line of (7) is larger. When the first term is larger, $\xi_{ia}(t) = 1$; when the second term is largest, $\xi_{ib}(t) = 1$, otherwise, $\xi_{ia}(t) = \xi_{ib}(t) = 0$.

The monopolist's HJB equation can be obtained similarly in terms of the behavior of the buyer:

$$w_m(\pi) = \sup_{p_{a,i},p_{b,i}} \left\{ \sum_{i=1}^{2} \Big(\xi_{ib}\left(\pi, p_a, p_b\right) p_{b,i} + \xi_{ia}\left(\pi, p_a, p_b\right) p_{a,i} + g\Big(\pi, h, \ell\Big) \frac{\xi_{ib}\left(\pi, p_a, p_b\right)}{2\rho\sigma_i^2} w_m''(\pi) \Big) \right\}. \quad (8)$$

Next, we explore the case of symmetric and asymmetric learning technologies. Under the monopolistic structure we are currently analyzing, the two cases lead to similar welfare conclusions, but we separate them because of the critical difference they feature under competition.

4.1.1 Symmetric buyers

The main takeaway of the symmetric case is that the monopolist who sells both products chooses which one to deliver to the market using the same belief threshold as in the welfare-maximizing benchmark, that is, $\pi_m^* = \pi_{\text{fb}}$. As a result, the monopoly achieves efficiency.

Proposition 3. *If $\sigma_1 = \sigma_2$, the revenue maximizing equilibrium is specified by the cutoff*

$$\pi_m^* = \pi_{\text{fb}}.$$

The above result is unsurprising because a monopolist with the power to make take-it-or-leave-it offers can extract all the surplus from symmetric buyers. But how robust is this result? Interestingly, we next show that it depends neither on symmetry nor the ability to price discriminate according to the buyers' learning technology.

4.1.2 Asymmetric Buyers

The following result shows that, under monopoly, efficiency is still achieved, even when buyers' learning technologies are heterogeneous.

Proposition 4. *If $\sigma_1 > \sigma_2$, the following holds under a monopolistic market structure. (i) There is a revenue-maximizing and efficient equilibrium with $p_{a,1}(\pi) = p_{a,2}(\pi)$ and $p_{b,1}(\pi) = p_{b,2}(\pi)$ for all $\pi \in (0,1)$. (ii) There is no efficient equilibrium in which $p_{a,1}(\pi) = p_{a,2}(\pi) = p_{b,1}(\pi) = p_{b,2}(\pi)$ for all $\pi \in (0,1)$.*

Importantly, part (i) of the proposition shows that the first-best is achieved with a pricing strategy that does not condition on the learning skill of the buyers, and thus does not require price discrimination. This is a key difference between asymmetries in the learning technologies and asymmetries in the valuation of the new product (i.e., heterogeneous parameters h and ℓ across buyers as in models a la Bergemann and Välimäki (1997)). In the latter case, it is well known that the incentive compatibility of the buyers induces inefficient revenue-maximizing allocations. In our model of asymmetric learning technologies, this does not happen.[18]

The intuition behind the result is as follows. The willingness to pay for product a is the same for both buyers and equal to μ_a. But their willingness to pay for product b at belief π potentially differs: it equals the instantaneous expected flow of utility $\mathbf{E}_\pi[\theta]$ plus the value of learning (i.e., v_i'') multiplied by the amount of information produced by the buyer. Even if $\mathbf{E}_\pi[\theta]$ is common across all the agents, differences in the learning components may create incentive compatibility issues. However, in our proof, we show that the monopolist can always obtain the total surplus by setting the price of the products equal to their expected utility flow. Indeed, when the monopolist uses this pricing strategy, all the agents have zero value of information (i.e., v'' is constantly zero) and, therefore, have the same willingness to pay, eliminating any incentive compatibility issue. However, it is an immediate consequence of Proposition 2 that if every type of price differentiation

[18] In a different setting Corrao, Flynn and Sastry (2023) shows a similar irrelevance for incentive compatibility when the agents are heterogenous in their attention costs. See also Bergemann and Välimäki (2010) for a general dynamic mechanism that allows the designer to achieve efficiency by charging their dynamic externalities.

is banned and the monopolist must use the same price for the two products, a distortion may arise.

Of course, other fairness concerns may arise since the inefficiency is eliminated in a monopolistic market, but the entire surplus accrues to the monopolist. Next, we show that if we introduce competition to obtain a more fair surplus division, efficiency is lost.

4.2 Main results: Competition

In this section, we present our main results. In the case of duopolistic competition between the sellers, the value function of seller $k \in \{a, b\}$ is the solution to the following HJB equation:

$$w_k(\pi) = \sup_{p_k} \left\{ \sum_{i=1}^{2} \xi_{ik}(\pi, p_k, p_{-k}(\pi)) p_{k,i} + g(\pi, h, \ell) \sum_{i=1}^{2} \frac{\xi_{ib}(\pi, p_k, p_{-k}(\pi))}{2\rho\sigma_i^2} w_k''(\pi) \right\}. \tag{9}$$

We next show that competitive symmetric and asymmetric markets will have very different welfare implications in sharp contrast to monopolistic market structures. Once again, we consider separately the case of symmetric and asymmetric buyers.

4.2.1 Symmetric buyers

When the buyers have the same learning technologies, and the equilibrium is symmetric, the HJB equation of seller k simplifies to:

$$w_k(\pi) = \sup_{p_k} \left\{ \underbrace{p_k \mathrm{Vol}_k(\pi, p_k, p_{-k})}_{\text{flow gain}} + \underbrace{\mathrm{Vol}_b(\pi, p_k, p_{-k}) \frac{g(\pi, h, \ell)}{2\rho\sigma^2} w_k''(\pi)}_{\text{learning gain from product } b} \right\}, \qquad k \in \{a, b\}, \tag{10}$$

where $\mathrm{Vol}_k(\pi, p_k, p_{-k}) = \sum_{i=1,2} \xi_{ik}(\pi, p_k, p_{-k})$ is the volume of seller k's sale.

The right-hand side of the HJB equation has two terms. The first is the expected flow payoff $p_k \mathrm{Vol}_k$ (given that the volume of seller k's sale is Vol_k), and the second is his continuation payoff that depends on Vol_b (i.e., the volume of seller b's sale) via $\mathrm{Vol}_b \frac{g(\pi,h,\ell)}{2\rho\sigma^2} w_k''(\pi)$.

We start with a preliminary caveat on equilibrium multiplicity.

Remark 1 (Equilibrium selection)**.** *Since we focus on the different efficiency properties of monopoly and competition, we want to consider the competition outcome with* ***minimal departure*** *from monopoly. As a monopoly is a situation in which the surplus accruing to the (unique) seller is maximal, the minimal departure is obtained by focusing on the equilibrium that maximizes sellers' profits. In what follows, we consider the equilibrium that is* ***most favorable to the sellers****. Since we will highlight the difference between monopoly and competition, our findings will be more surprising the less we depart from the monopoly with our equilibrium selection.*

Note that this equilibrium selection differs from the one imposed by Bergemann and Välimäki (2000). Notably, both equilibria satisfy the "cautious" refinement criterion proposed by Bergemann and Välimäki (2000) requiring sellers not to make offers they would not like to be accepted. But caution alone does not pin down the equilibrium. Bergemann and Välimäki (2000) resolve this multiplicity by imposing smooth pasting in the value function of seller b (rather than seller a). Instead, we select the equilibrium that is more favorable for both sellers to depart minimally from the monopoly. We show in Lemma 3 in the Appendix that smooth pasting for the seller a (rather than b) must be satisfied to maximize values for both sellers.

Next, we characterize the seller's profit-maximizing pricing strategy in a symmetric cutoff equilibrium.

Symmetric buyers choose the same strategy. When all other buyers purchase the product a, the individual buyer's value function satisfies the following HJB equation:

$$v = \max\Big\{\mu_a - p_a, \mathbf{E}_\pi[\theta] - p_b + \frac{1}{2\rho\sigma^2} g v'', 0\Big\}. \tag{11}$$

When all other buyers purchase the product b, v satisfies

$$v = \max\Big\{\mu_a - p_a + \frac{1}{2\rho\sigma^2} g v'', \mathbf{E}_\pi[\theta] - p_b + \frac{1}{\rho\sigma^2} g v'', 0\Big\}. \tag{12}$$

In either cases, the indifference conditions between buying product a and b is

$$p_b - p_a = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma^2} g v''. \tag{13}$$

The difference in prices is the difference between the expected utility net of the information cost of exploring the new product b.

When all buyers purchase the product a or b, the seller a's value function satisfies the following HJB equation:

$$w_a = \max\left\{2p_a, \frac{1}{\rho\sigma^2} g w_a''\right\}. \tag{14}$$

The right-hand side of the previous equation indicates two options for the seller a: sell to all buyers with the price p_a or gain from learning by conceding the market to seller b. The seller b's value function satisfies

$$w_b = \max\left\{0, p_b + \frac{1}{\rho\sigma^2} g w_b''\right\}. \tag{15}$$

The right-hand side indicates two options for seller b: concede the market to seller a and obtain zero profit, or sell to all buyers with the price p_b and gain from the associated learning.

The sellers' indifference conditions characterizes the threshold π^* where the seller a concede the whole market to the seller b.

$$\mathbf{E}_{\pi^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma^2} g\Big(v'' + w_a'' + w_b''\Big)(\pi^*) = 0. \tag{16}$$

To see this, we consider two cases below.

(i) When the left-hand side of (16) < 0, the seller a and b choose the prices

$$p_a = \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma^2} g\Big(v'' + w_b''\Big) \quad \text{and} \quad p_b = -\frac{1}{2\rho\sigma^2} g w_b''.$$

The choice of p_b and (15) show that the seller b is willing to concede the whole market to the seller a, the seller a can charge the highest price so that the buyer's indifference condition (13) still holds. When the left-hand side of (16) < 0, p_a satisfies

$$2p_a > \frac{1}{\rho\sigma^2} g w_a'',$$

so that the seller a is willing to sell to the whole market with the price p_a.

(ii) When the left-hand side of (16) > 0, the seller a and b choose the prices

$$p_a = \frac{1}{2\rho\sigma^2} g w_a'' \quad \text{and} \quad p_b = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma^2} g\Big(v'' + w_a''\Big).$$

The choice of p_a and (14) show that the seller a is willing to concede the whole market to the seller b, the seller b can charge the highest price so that the buyer's indifference condition (13) still holds. When the left-hand side of (16) > 0, p_b satisfies

$$2p_b + \frac{1}{\rho\sigma^2} g w_b'' > 0,$$

so that the seller b is willing to sell to the whole market with the price p_b.

To obtain the welfare implication, let us derive the equation that total welfare $w_a + w_b + 2v$ satisfies. When $\pi < \pi^*$, the seller a dominates the whole market, we sum (11), (14), and (15) to obtain

$$w_a + w_b + 2v = 2\mu_a.$$

When $\pi > \pi^*$, the seller b dominates the whole market, we sum (12), (14), and (15) to obtain

$$w_a + w_b + 2v = 2\mathbf{E}_\pi[\theta] + \frac{1}{\rho\sigma^2} g\Big(w_a'' + w_b'' + 2v''\Big).$$

The value matching at π^* and the previous two equations imply that

$$\mathbf{E}_{\pi^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma^2} g\Big(w_a'' + w_b'' + 2v''\Big)(\pi^*) = 0. \tag{17}$$

Comparing (16) and (17), we must have

$$v''(\pi^*) = 0. \tag{18}$$

In other words, no information externality exists for buyers at the threshold π^*.

Thanks to (18), we have from (16) that

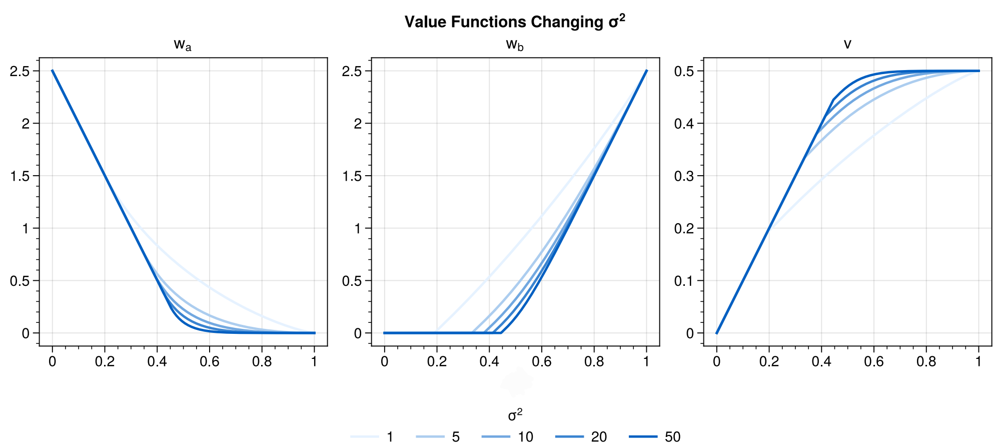
$$\mathbf{E}_{\pi^*}[\theta] + \frac{1}{2\rho\sigma^2} g\Big(2v'' + w_a'' + w_b''\Big)(\pi^*) = \mu_a.$$

This is exactly the first order condition for the welfare maximization problem: at π^{fb}, the expected utility of product b plus the information gain is exactly the utility of product a. Therefore

$$\pi^* = \pi^{\mathrm{fb}}.$$

Theorem 1. *The following holds. (i) The cutoff equilibrium with the highest sellers' profits is efficient (i.e., welfare-maximizing) and has a unique cutoff π^*. (ii) The consumer surplus is strictly higher than under a monopoly. (iii) The value function of the two sellers is convex, and the value function of the buyers is concave.*

The Appendix spells out all the details of this derivation and shows that in the highest sellers' profits equilibrium, the conjectured equation (16) must hold.



(a) This figure plots the value functions w_a, w_b and v when σ^2 changes. As σ^2 increases, the cutoff π^* moves to the right. Competition equilibrium with symmetric buyers. Other parameters $h = 1, \ell = 0, \mu_a = .5$ (Explicit characterizations of the value functions are provided in the proof of Theorem 1.)

It is important to understand the economic intuition behind the result. Initially, it does not seem surprising. Standard reasoning from static markets tells us that the sellers have no reason to price discriminate since the buyers are symmetric. In static markets, it is well known that the absence of incentives to price discriminate (or the possibility for the seller to perfectly discriminate) is sufficient to guarantee that the seller's market power does not induce inefficiencies. One may think that the same is happening here. Our next result below shows that this is not so: if buyers are asymmetric, the possibility of price discrimination does not amend inefficiencies. Indeed, in markets with learning externalities, efficiency is obtained only if the seller also can internalize the learning externality of the other market participants. The proof of Theorem 1 shows that this is the case when buyers are symmetric. However, when buyers are asymmetric in their learning technologies, the next section shows that the externality is not internalized, resulting in inefficiency.

4.2.2 Asymmetric buyers

First, we present the result that the decentralized outcome induced by competition is no longer efficient with asymmetric buyers, and we then explore the nature of the inefficiency.

Theorem 2. *If $\sigma_1 > \sigma_2$, the equilibrium with the highest sellers' profit is inefficient. However, there is efficiency for the top learners:*

$$\pi_2^* = \pi_{\mathrm{fb},2}.$$

To understand the intuition of this result, we start with the buyers' problem. The HJB equation for the buyers is almost the same as in the symmetric case, with the only difference that the learning component involved in the trade-off between the two products is now buyer-specific:

$$v_i(\pi) = \max\Bigg\{\mu_a - p_{a,i}(\pi) + g(\pi,h,\ell)\sum_{j\neq i}\frac{\xi_{jb}(\pi,p(\pi))}{2\rho\sigma_j^2}v_i''(\pi),$$
$$\mathbf{E}_\pi[\mu_b] - p_{b,i}(\pi) + g(\pi,h,\ell)\sum_{j=1}^{2}\frac{\xi_{jb}(\pi,p(\pi))}{2\rho\sigma_j^2}v_i''(\pi)\Bigg\}.$$

Price competition between sellers imposes indifference between the arguments of the above maximization. Indeed, in the right-hand side of the above HJB, if the second argument were larger, a profitable deviation for seller b would be to slightly increase $p_{b,i}$, collecting higher per-unit revenues and selling to the same number of buyers. An analogous profitable deviation would obtain for the seller a if the first argument were strictly larger than the second. Therefore the indifference conditions for buyers are

$$p_{b,1}(\pi) - p_{a,1}(\pi) = \mathbf{E}_\pi\left[\theta\right] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) v_1''(\pi), \tag{19}$$

for buyer 1 and

$$p_{b,2}(\pi) - p_{a,2}(\pi) = \mathbf{E}_\pi\left[\theta\right] - \mu_a + \frac{1}{2\rho\sigma_2^2} g(\pi, h, \ell) v_2''(\pi), \tag{20}$$

for buyer 2. As we will show later, both v_1 and v_2 are concave. Thus equations (19) and (20) together give us the rebate that seller b awards the buyers to compensate them for their exploration.

Plugging the indifference conditions of the buyers into the HJB equations of sellers, we show that seller a's and b's problems can be characterized by the following HJB equations:

$$\begin{aligned} w_a(\pi) = &\max\left\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2} g(\pi, h, \ell)\Big(v_2''(\pi) + w_b''(\pi)\Big), \frac{1}{2\rho\sigma_2^2} g(\pi, h, \ell) w_a''(\pi)\right\} \\ &+ \max\left\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell)\Big(v_1''(\pi) + w_b''(\pi)\Big), \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) w_a''(\pi)\right\}, \end{aligned} \tag{21}$$

$$\begin{aligned} w_b(\pi) = &\max\left\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g(\pi, h, \ell)\Big(v_2''(\pi) + w_a''(\pi)\Big) + \frac{1}{2\rho\sigma_2^2} g(\pi, h, \ell) w_b''(\pi), 0\right\} \\ &+ \max\left\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell)\Big(v_1''(\pi) + w_a''(\pi)) + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) w_b''(\pi), 0\right\}. \end{aligned} \tag{22}$$

In each maximization of (21) and(22), when the first term is larger than the second, then a product is sold to a corresponding agent. For example, when $\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_b'') > \frac{1}{2\rho\sigma_2^2} g w_a''$, it is better for the seller to sell the product a to buyer 2 with the price $\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_b'')$ than let b sell the product b to buyer 2, in which case, seller a

value is $\frac{1}{2\rho\sigma_2^2} g w_a''$.

We conjecture that the cutoff π_2^* is pinned down by

$$\mathbf{E}_{\pi^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g(\pi_2^*, h, \ell)\Big(v_2''(\pi_2^*) + w_a''(\pi_2^*)\Big) + \frac{1}{2\rho\sigma_2^2} g(\pi_2^*, h, \ell) w_b''(\pi_2^*) = 0, \tag{23}$$

where both seller a and b are indifferent to sell to buyer 2. Meanwhile π_1^* is pinned down by

$$\mathbf{E}_{\pi^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_1^*, h, \ell)\Big(v_1''(\pi_1^*) + w_a''(\pi_1^*)\Big) + \frac{1}{2\rho\sigma_1^2} g(\pi_1^*, h, \ell) w_b''(\pi_1^*) = 0, \tag{24}$$

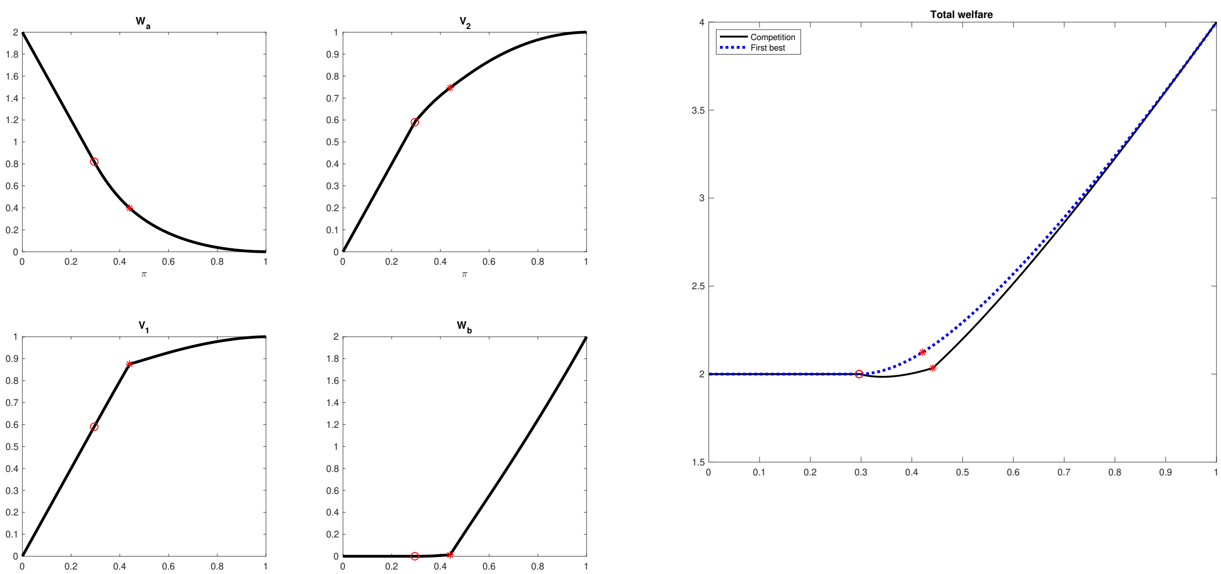
where both seller a and b are indifferent to sell to buyer 1.

When $\pi < \pi_2^*$, both the left-hand side (LHS) of (23) < 0 and the LHS of (24) < 0. Seller a is willing to sell to both buyers and seller b stays out of the market. When $\pi \in (\pi_2^*, \pi_1^*)$, the LHS of (23) > 0 and the LHS of (24) < 0. Seller a is only willing to sell to buyer 1, and seller b is only willing to sell to buyer 2. When $\pi > \pi_1^*$, both the LHS of (23) > 0 and the LHS of (24) > 0. Seller b is willing to sell to both buyers.

Figure 3 presents a numeric example. Panel (a) presents the value functions w_a, w_b, v_1, and v_2. The plots show that w_a and w_b are convex and v_1 and v_2 are concave. w_b is strictly positive in the region (π_2^*, π_1^*), even though its value is close to zero in this example. Panel (b) compares the welfare between the first-best case and the equilibrium with competition and asymmetric buyers. We see that the first-best welfare is higher than that in equilibrium. Moreover, $\pi_{\mathrm{fb},2} = \pi_2^*$, but $\pi_{\mathrm{fb},1} < \pi_1^*$. In this example, the conjectures in (23) and (24) are verified in Figure 4. The same qualitative results hold with different parameter values.

The inefficiency at π_1^* (with respect to $\pi_{\mathrm{fb},1}$) follows from not considering the learning externality that consumption by the worst learner induces over the best learner. Indeed, when seller b serves a buyer with the bad learning technology, an informational impact is produced for every market participant. The profit-maximizing price-setting by seller b internalizes the learning impact for him and the buyer, and competition incorporates the learning externality of seller a into the price. However, the learning externality for the best learner is not internalized.

This can also be understood from equation (24). This equation represents the indifference condition for sellers a and b to sell to buyer 1. It does not incorporate information



(a) Value functions w_a, w_b, v_1, and v_2. The horizontal coordinate of the red circle is π_2^*, and the horizontal of the red asterisk is π_1^*.

(b) Welfare in the first-best and equilibrium with competition and asymmetric buyers. Horizontal coordinate of the red circle is $\pi_{\text{fb},2} = \pi_2^*$. The horizontal coordinate of the bottom red asterisk is π_1^*. The horizontal coordinate of the top red asterisk is $\pi_{\text{fb},1}$.

Figure 3: Equilibrium with competition and asymmetric buyers. Parameters are $h = 2$, $\ell = 0$, $\mu_a = 1$, $\sigma_1 = 4$, $\sigma_2 = 3$, and $\rho = 0.5$.

from buyer 2, who is already using the new product. This pricing between sellers a, b, and buyer 1 creates an externality to buyer 2. Indeed, under the first-best maximization, the welfare-maximizing condition imposes

$$\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell)\Big(w_a'' + w_b'' + v_1'' + v_2''\Big)(\pi_{\text{fb},1}) = 0.$$

However, our numeric example in Figure 3 shows that

$$\lim_{\pi \uparrow \pi_1^*} v_2''(\pi) < 0.$$

Therefore, there is a negative information externality to buyer 2 when buyer 1 switches

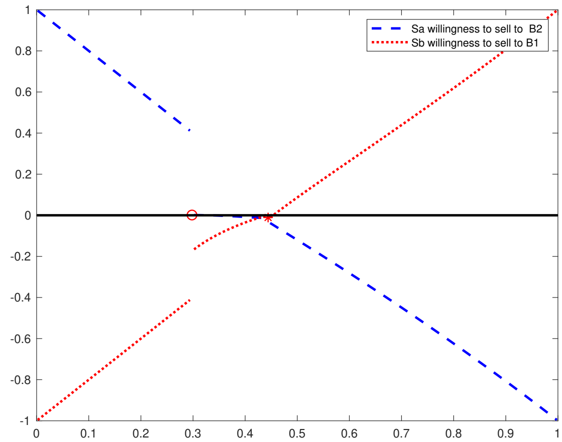


Figure 4: Seller indifference conditions. The blue dashed line plots the negative of the left-hand side of (23). When it is positive, seller a is willing to sell to buyer 2. The red dotted line represents the left-hand side of (24). When it is positive, seller b is willing to sell to buyer 1. These two lines cross zero at π_2^* and π_1^* respectively, confirming numerically (23) and (24).

from product a to b. The intuition is that more differentiation between the sellers' qualities allows them to extract higher profits. Combined with (24) and the previous equation, we conclude that $\pi_1^* \neq \pi_{\text{fb},1}$.

Meanwhile, for the lower threshold, the welfare-maximizing condition in the first-best problem implies

$$\mathbf{E}_{\pi_{\text{fb},2}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g(\pi_{\text{fb},2}, h, \ell)\Big(w_a'' + w_b'' + v_1'' + v_2''\Big)(\pi_{\text{fb},2}) = 0.$$

We prove in Theorem 2 that

$$v_1''(\pi_2^*) = 0.$$

Therefore, there is no information externality to buyer 1 when buyer 2 starts to explore the uncertain product. Combined with (23) and the previous equation, we conclude that $\pi_2^* = \pi_{\text{fb},2}$.

The externality issue does not arise in the symmetric case because, at the unique threshold (the same for the smallest threshold π_2^* here), no buyer has a strictly positive value of information i.e., $v''(\pi_2^*) = 0$. Indeed, around the unique threshold in the homogeneous case (and the lowest threshold of the heterogeneous one), the market agents are almost certain to receive a string of bad signals that eliminates experimentation with good b, making the information previously generated useless.

The efficiency for the top learners' part of the result highlights the way in which distortions arise. Competition does not affect the number and quality of the innovations that are given a chance, that is, the threshold for being tested by the fraction of the market that produces better (more precise) information about the product (i.e., to start the beta phase). However, what is affected is the confidence in the product's quality required to start serving the entire market.

In the final portion of this section, we highlight that the extent of inefficiency is not monotone in the difference between the learning technologies of buyers. Theorem 1 already guarantees that efficiency holds in the case of symmetric buyers, and the next result shows that the distortion disappears also in the limit, where one of the two buyers does not produce any valuable information (through his experience) about the product's quality.

Proposition 5. *Fix σ_2. The equilibrium with a buyer that does not generate any information about the quality of the product is efficient:*

$$\lim_{\sigma_1 \to \infty} (\pi_1 - \pi_{\text{fb},1}) = 0$$

The intuition behind the previous result is that, if serving the general public does not produce additional information about the product, there is no learning externality to consider when deciding whether the product is ready for the entire market.

Finally, note that the nonmonotonicity in the difference in learning technologies implied by the previous proposition is a robust feature of the model, and it continues to arise even with multiple levels of learning technologies. However, what is lost in the more general case is the stark conclusion that, if the worst learners become completely uninformative, then efficiency is fully restored. Indeed, competition induces some inefficiency as long as there are two types of learners with a variance of the signal strictly between zero and plus infinity.

5 Discussion

In this section, we first demonstrate that efficiency can be restored if the seller has the ability to provide multilateral contracts. Following that, we investigate the determinants and comparative statics of the transition period, the beta phase, which is unique to our model featuring asymmetries in learning technologies.

5.1 Multilateral Contracts

Even though competition introduces inefficiency to the market with asymmetric buyers, there is a way, however, to eliminate the distortion and maintain competition. This is achieved by increasing the commitment power of the sellers. More precisely, suppose that now seller $k \in \{a, b\}$ can commit to offering a multilateral contract of the following form: one-unit of product k will be delivered to buyer $i \in \{1, 2\}$ if buyer i makes a transfer $t^i_{k,i}$ to seller k and buyer $j \neq i$ makes a transfer $t^j_{k,i}$ to seller k.

That is, we allow the seller to ask a buyer to pay or be compensated for the product being delivered to another consumer. We notice that the buyer who does not receive the product may benefit from the transfer because of the learning externality that is generated by the use of product b. More precisely, we set

$$t^2_{a,1}(\pi) = -v''_2(\pi)\frac{g(\pi, h, \ell)}{2\rho\sigma_1^2}, \qquad t^1_{a,2}(\pi) = -v''_1(\pi)\frac{g(\pi, h, \ell)}{2\rho\sigma_2^2},$$
$$t^2_{b,1}(\pi) = v''_2(\pi)\frac{g(\pi, h, \ell)}{2\rho\sigma_1^2}, \qquad t^1_{b,2}(\pi) = v''_1(\pi)\frac{g(\pi, h, \ell)}{2\rho\sigma_2^2}.$$

Because v_1 and v_2 are concave, when seller b sells to buyer 1, he also compensates buyer 2 the amount $-v''_2(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2}$, which is the externality cost to buyer 2. When seller a sells to buyer 1 instead, buyer 2 transfers to seller a the amount $-v''_2(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2}$ which is the externality cost he would shoulder if seller b sells to buyer 1 instead.

Although there are service markets in which a similar structure may be implemented in the form of a subscription to a platform that shares buyers' experiences, we think that, in most cases, assuming such commitment power is unwarranted. Still, if we allow for this possibility, the competition outcome becomes efficient.

Proposition 6. *When sellers are allowed to use multilateral contracts, the equilibrium is efficient.*

5.2 Beta phase

We consider two given thresholds, $\pi_1 > \pi_2$. Therefore insights in this section apply both to the case of competition and the first-best.

In the beta phase, only the customers with better learning technologies buy the new product. When the public belief becomes sufficiently optimistic (i.e., $\pi > \pi_1$), all buyers purchase the new product; when the public belief is sufficiently pessimistic (i.e., π reaches π_2, π stays there forever), no buyers purchase the new product in the future and it fails. In the following proposition, we explicitly characterize the expected length of the beta phase and the probability of new product failure or full adoption. Let $\tau = \inf\{t : \pi_t \notin (\pi_2, \pi_1)\}$ denote the time needed to exit the beta phase.

Proposition 7. *Let $\pi_2 < \pi_0 < \pi_1$. Define $\sigma(y) := \frac{y(1-y)(h-\ell)}{\sigma_2}$. Then,*

$$\mathbf{E}_{\pi_0}[\tau] = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2} \int_{\pi_2}^{\pi_0} (y - \pi_2) \frac{2dy}{\sigma^2(y)} + \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2} \int_{\pi_0}^{\pi_1} (\pi_1 - y) \frac{2dy}{\sigma^2(y)}.$$

Particularly, the expected length of the beta phase is strictly increasing (decreasing) in the initial opinion for sufficiently low (high) initial opinions π_0.
Moreover, the following hold.

- *The probability of discarding the new product as a failure is $Pr_{\pi_0}\{\text{discarding}\} = Pr_{\pi_0}\{\pi_\tau = \pi_2\} = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2}$.*
- *The probability that the new product serves the whole market is $Pr_{\pi_0}\{\textit{full adoption}\} = Pr_{\pi_0}\{\pi_\tau = \pi_1\} = \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2}$.*

Particularly, $\partial_{\pi_0} Pr_{\pi_0}\{\text{discarding}\} < 0$, and $\partial_{\pi_0} Pr_{\pi_0}\{\textit{full adoption}\} > 0$.

The above result explicitly characterizes the expected length of the beta phase and the probabilities that the new product either serves the whole market or is discarded as a failure in terms of the endogenous thresholds π_1 and π_2. Moreover, it produces intuitive comparative statics based on the initial market belief about the new product. In particular, the expected length of the beta phase increases with the initial market belief when this belief is initially sufficiently small (i.e., $\lim_{\pi_0 \searrow \pi_2} \partial_{\pi_0} \mathbf{E}_{\pi_0}$[beta phase] > 0), and the expected length of the beta phase decreases with the initial market belief when this belief is initially sufficiently large (i.e., $\lim_{\pi_0 \searrow \pi_2} \partial_{\pi_0} \mathbf{E}_{\pi_0}$[beta phase] > 0). Moreover, the probability of discarding the new product as a failure decreases in the initial market belief, and the probability that the new product starts to serve the whole market increases in the initial market belief.

6 Conclusion

We investigate the interplay between market structure (monopoly versus duopoly) and asymmetry in learning technologies in a dynamic product market. In this market, a new product of unknown quality competes with an established one, and public information on the unknown quality evolves dynamically through Bayesian learning. We determine that the optimal policy is characterized by a series of belief thresholds, including a beta phase during which only the best learners explore the new product.

We examine the efficiency implications of different market structures. Under a monopolistic market structure where the same seller offers both the new and the old product, equilibrium is always efficient. In stark contrast, when two sellers compete to market their respective products, efficiency is only achieved if buyers possess symmetric learning technologies.

We identify the inefficiency as a learning externality generated by one buyer's product consumption for other buyers. The equilibrium inefficiency exhibits two features: (i) efficiency for top learners (i.e., the threshold for initiating service to the best learners or entering the beta phase, remains the efficient one), and (ii) nonmonotonicity (i.e., distortions are not monotone in the extent of the asymmetry).

We demonstrate that the distortion vanishes in markets where sellers can utilize multilateral contracts. Additionally, we analyze the learning progression in market belief and the expected time until the new product either fails or is fully adopted by the entire market.

7 Appendix

Proof of Proposition 1. Restatement of Theorem 1 in Bergemann and Välimäki (2000). □

Before proving Proposition 2, we first present a verification result, proved in the Online Appendix. For notational convenience, we set $\pi_{\text{fb},0} = 1$, $\pi_{\text{fb},3} = 0$, $\sigma_0^2 = \infty$, and $\sigma_3^2 = 0$.

Proposition 8. *Suppose that there exist $\pi_{\text{fb},1}, \pi_{\text{fb},2} \in (0,1)$, $\pi_{\text{fb},1} > \pi_{\text{fb},2}$ and a bounded convex function W_{avg} which satisfies the following conditions:*

(i) W_{avg} is twice continuously differentiable in $(0, \pi_{\text{fb},2})$, $(\pi_{\text{fb},2}, \pi_{\text{fb},1})$, $(\pi_{\text{fb},1}, 1)$, and for $i \in \{0,1,2\}$ satisfies

$$W_{avg}(\pi) = \mu_a + \frac{1}{2}\sum_{j=i+1}^{2}\left(\mathbf{E}_\pi[\theta] - \mu_a\right) + \frac{1}{2\rho} g(\pi,h,\ell) W''_{avg}(\pi) \sum_{j=i+1}^{2} \frac{1}{\sigma_j^2}, \quad \pi \in (\pi_{\text{fb},i+1}, \pi_{\text{fb},i}); \tag{25}$$

(ii) W_{avg} satisfy value matching and smooth pasting conditions

$$\lim_{\pi \uparrow \pi_{\text{fb},i}} W_{avg}(\pi) = \lim_{\pi \downarrow \pi_{\text{fb},i}} W_{avg}(\pi), \tag{26}$$

$$\lim_{\pi \uparrow \pi_{\text{fb},i}} W'_{avg}(\pi) = \lim_{\pi \downarrow \pi_{\text{fb},i}} W'_{avg}(\pi), \quad \forall i \in \{1,2\}; \tag{27}$$

(iii) $\frac{1}{\rho\sigma_{i+1}^2} g(\pi,h,\ell) W''_{avg}(\pi) > \mu_a - \mathbf{E}_\pi[\theta] > \frac{1}{\rho\sigma_i^2} g(\pi,h,\ell) W''_{avg}(\pi)$, for every $\pi \in (\pi_{\text{fb},i+1}, \pi_{\text{fb},i})$ and $i \in \{0,1,2\}$.

Then, W_{avg} is the first best value function, and a first best optimal strategy is given in (6).

We now construct a function W_{avg} satisfying conditions in Proposition 8 to prove Proposition 2.

Proof of Proposition 2. First, we consider (25) in the three regions created by the posited cutoffs and solve them individually using the Wronskian approach of second-order ODEs (Zaitsev and Polyanin (2002)). In this approach, solutions of second-order ODEs are represented as linear combinations of general solutions, and boundary conditions identify coefficients.

- Case 1: If $\pi \in (\pi_{\text{fb},2}, \pi_{\text{fb},1})$ then

$$W_{avg,1}(\pi) = \frac{1}{2}\Big(\mu_a + \mathbf{E}[\mu_b]\Big) + \zeta_1 \pi^{\frac{1}{2}(\lambda+1)}(1-\pi)^{-\frac{1}{2}(\lambda-1)} + \zeta_2 \pi^{-\frac{1}{2}(\lambda-1)}(1-\pi)^{\frac{1}{2}(\lambda+1)}$$

  with $\lambda = \sqrt{1 + \frac{8\rho\sigma_2^2}{(h-\ell)^2}}$.

- Case 2: If $\pi \geq \pi_{\text{fb},1}$, the general solution has the form

$$W_{avg,0}(\pi) = \mathbf{E}_\pi[\theta] + \zeta^* \pi^{\frac{1}{2}(\bar{\lambda}+1)}(1-\pi)^{-\frac{1}{2}(\bar{\lambda}-1)} + \zeta_0 \pi^{-\frac{1}{2}(\bar{\lambda}-1)}(1-\pi)^{\frac{1}{2}(\bar{\lambda}+1)}$$

  with $\bar{\lambda} = \sqrt{1 + \frac{8\rho\sigma_1^2\sigma_2^2}{(h-\ell)^2\left(\sigma_1^2+\sigma_2^2\right)}}$. However, note that W_{avg} is bounded by h on $\pi \geq \pi_{\text{fb},1}$, thus $\zeta^* = 0$ as $\lim_{\pi\to 1}(1-\pi)^{\frac{1}{2}(1-\lambda)}$ explodes in this region: $W_{avg,0}(\pi) = \mathbf{E}_\pi[\theta] + \zeta_0 \pi^{-\frac{1}{2}(\bar{\lambda}-1)}(1-\pi)^{\frac{1}{2}(\bar{\lambda}+1)}$.

- Case 3: If $\pi < \pi_{\text{fb},2}$, we trivially have $W_{avg}(\pi) = \mu_a$.

There are five unknowns $\zeta_0, \zeta_1, \zeta_2, \pi_{\text{fb},1}, \pi_{\text{fb},2}$ that we need to identify W_{avg} on $(0,1)$. Therefore, we are going to use five conditions. We already have four conditions in (26) and (27). The fifth equation will be a super-contact condition. The next claim, proved in the Online Appendix, establishes its necessity.

Claim 1. *The value matching and Proposition 8 (iii) imply that*

$$W''_{avg,1}(\pi_{\text{fb},1}) = W''_{avg,0}(\pi_{\text{fb},1}). \tag{28}$$

With these five conditions, we pin down the five unknowns, hence W_{avg}, which is identified as $W_{avg,0}, W_{avg,1}$, or μ_a on corresponding intervals, satisfies Proposition 8 (i) and (ii). The convexity of W_{avg} is verified numerically.

To show that W_{avg} constructed satisfies Proposition 8 (iii), we first note that (28) together with (70) and (71) later implies that

$$\mathbf{E}_{\pi_{\mathrm{fb},1}}[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi_{\mathrm{fb},1}, h, \ell) W''_{avg}(\pi_{\mathrm{fb},1}) = 0.$$

Moreover, from the value matching at $\pi_{\mathrm{fb},2}$, we have

$$\mathbf{E}_{\pi_{\mathrm{fb},2}}[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi_{\mathrm{fb},2}, h, \ell) W''_{avg}(\pi_{\mathrm{fb},2}+) = 0,$$

where $W''_{avg}(\pi_{\mathrm{fb},2}+) = \lim_{\pi \downarrow \pi_{\mathrm{fb},2}} W''_{avg}(\pi)$. The convexity of W_{avg} and the two equations above imply that $\pi_{\mathrm{fb},1}, \pi_{\mathrm{fb},2} < \pi_{\mathrm{myoptic}}$. Therefore Proposition 8 (iii) is satisfied in $(0, \pi_{\mathrm{fb},2})$, because $W''_{avg} = 0$ there.

We claim that $\mathbf{E}_{\pi}[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi, h, \ell) W''_{avg}(\pi) > 0$ when $\pi \in (\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$. If not, there exists $\tilde{\pi} \in (\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$ such that $\mathbf{E}_{\tilde{\pi}}[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\tilde{\pi}, h, \ell) W''_{avg}(\tilde{\pi}) \leq 0$. By the equation satisfied by W_{avg} in $(\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$ (see (25)), we have $W_{avg}(\tilde{\pi}) \leq \mu_a$. This contradicts with the fact that W_{avg} is an increasing function in $(\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$, because $W_{avg}(\pi_{\mathrm{fb},2}) = \mu_a$, $W'_{avg}(\pi_{\mathrm{fb},2}) = 0$, and W_{avg} is convex.

We claim that $\mathbf{E}_{\pi}[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi, h, \ell) W''_{avg}(\pi) < 0$ when $\pi \in (\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$. If not, there exists $\tilde{\pi} \in (\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$ such that $\mathbf{E}_{\tilde{\pi}}[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\tilde{\pi}, h, \ell) W''_{avg}(\tilde{\pi}) = 0$. Then W_{avg} would satisfy its equation in $(\pi_{\mathrm{fb},2}, 1)$. This contradicts with the assumption that $\tilde{\pi} < \pi_{\mathrm{fb},1}$. The inequality $\mathbf{E}_{\pi}[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi, h, \ell) W''_{avg}(\pi) > 0$ when $\pi \in (\pi_{\mathrm{fb},1}, 1)$ can be proved similarly.

This concludes the proof of Proposition 2. □

To prepare for the proof of Theorem 1, we first prove the following result regarding the prices in a symmetric cutoff equilibrium.

Lemma 2. *The prices are as follows in a symmetric cutoff equilibrium with the highest sellers' profits.*

If $\pi \leq \pi^*$:

$$p_a(\pi) = \mu_a - \mathbf{E}_{\pi}[\theta] \qquad \text{and} \qquad p_b(\pi) = 0. \tag{29}$$

If $\pi > pi^*$:

$$p_a(\pi) = \frac{g(\pi,h,\ell)}{2\rho\sigma^2} w_a''(\pi) \qquad \textit{and} \qquad p_b(\pi) = \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{2\rho\sigma^2}\Big(v''(\pi) + w_a''(\pi)\Big). \tag{30}$$

Proof. To understand this pricing rule, let us first focus on the $\pi > \pi^*$ case, where all buyers purchase from seller b. The buyer's HJB equation (7) implies that

$$\mu_a - p_a(\pi) + \frac{g(\pi,h,\ell)}{2\rho\sigma^2} v''(\pi) \leq \mathbf{E}_\pi[\theta] - p_b(\pi) + \frac{g(\pi,h,\ell)}{\rho\sigma^2} v''(\pi), \tag{31}$$

for $\pi > \pi^*$. At the equilibrium, due to price competition between sellers, (31) holds with equality. Indeed, if the right-hand side was larger, it would be profitable for seller b to slightly increase p_b, collecting higher per unit revenues and selling to the same number of buyers. As a result, we must have

$$\underbrace{p_a(\pi) - p_b(\pi)}_{\text{Price difference}} = \underbrace{\mu_a - \mathbf{E}_\pi[\theta]}_{\text{Utility difference}} \underbrace{-\frac{g(\pi,h,\ell)}{2\rho\sigma^2} v''(\pi)}_{\text{Information value}}. \tag{32}$$

The difference in prices is the sum between the utility difference and an information value. We will show later that v is concave. Therefore the information value is positive, which widens the price difference and compensates buyers to use the new product b by lowering its price.

Meanwhile, given all buyers purchase from b when $\pi > \pi^*$, seller a's HJB equation (9) implies that

$$w_a(\pi) = \frac{g(\pi,h,\ell)}{\rho\sigma^2} w_a''(\pi) \geq 2p_a(\pi), \tag{33}$$

where the inequality means decreasing the price p_a to win over all buyers is suboptimal, compared to letting all buyers purchase from seller b and collecting information gain

$\frac{g(\pi,h,\ell)}{\rho\sigma^2}w_a''(\pi)$ instead. Similarly, seller b's HJB equation (9) implies that

$$w_b(\pi) = 2p_b(\pi) + \frac{g(\pi,h,\ell)}{\rho\sigma^2}w_b''(\pi) \geq 0, \tag{34}$$

where the inequality means selling to all buyers is better than the alternative of not selling at all.

Combining the indifference condition (32) and the inequalities in (33) and (34), we obtain admissible intervals where equilibrium prices must reside:

$$p_a(\pi) \in \Big[\mu_a - \mathbf{E}_\pi[\theta] - \frac{g(\pi,h,\ell)}{2\rho\sigma^2}\big(v''(\pi) + w_b''(\pi)\big),\ \frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_a''(\pi)\Big], \tag{35}$$

$$p_b(\pi) \in \Big[-\frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_b''(\pi),\ \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{2\rho\sigma^2}\big(v''(\pi) + w_a''(\pi)\big)\Big]. \tag{36}$$

Choosing p_a and p_b as the high end of their admissible intervals, the profit for seller b is maximized, and we obtain the pricing rule in (30).

When $\pi < \pi^*$, an argument similar to those above yields the admissible intervals for prices, similar to (35) and (36) but with their high and low ends switched. Then the pricing rule (29) follows because there is no learning when $\pi < \pi^*$, hence v, w_a, and w_b are all linear functions. □

Proof of Theorem 1. For $\pi > \pi^*$, it follows from (10) that

$$w_a(\pi) = \frac{g(\pi,h,\ell)}{\rho\sigma^2}w_a''(\pi). \tag{37}$$

Using Wronskian approach to solve second-order ODEs (Zaitsev and Polyanin (2002)), we have $w_a(\pi) = \zeta_1\left[\pi^{\frac{1}{2}(1-\lambda)}(1-\pi)^{\frac{1}{2}(1+\lambda)}\right] + \zeta_2\left[\pi^{\frac{1}{2}(1+\lambda)}(1-\pi)^{\frac{1}{2}(1-\lambda)}\right]$, where $\lambda = \sqrt{1 + 4\frac{\sigma^2\rho}{(h-\ell)^2}}$, ζ_1 and ζ_2 are two costants to be determined.

Because w_a is bounded, $\zeta_2 = 0$, otherwise the second term on the right-hand side of the expression for $w_a(\pi)$ above explodes as $\pi \to 1$. Hence,

$$w_a(\pi) = \zeta_1\left[\pi^{\frac{1}{2}(1-\lambda)}(1-\pi)^{\frac{1}{2}(1+\lambda)}\right], \tag{38}$$

for some positive constant ζ_1 to be determined.

To pick an equilibrium that maximizes the sellers' value, we prepare the following important result.

Lemma 3. *Given p_a as in Lemma 2, the equilibrium in which sellers' value is maximized is pinned down by the value matching $\lim_{\pi\uparrow\pi^*} w_a(\pi) = \lim_{\pi\downarrow\pi^*} w_a(\pi)$, and the smooth pasting condition $\lim_{\pi\uparrow\pi^*} w_a'(\pi) = \lim_{\pi\downarrow\pi^*} w_a'(\pi)$ for some threshold level π^*.*

For the rest of the proof of Theorem 1, we use the value matching and smooth pasting condition to pin down π^* and functions v, w_a, and w_b.

First, the value matching and smooth pasting for seller a at π^* give

$$w_a(\pi^*) = 2\Big(\mu_a - \mathbf{E}_{\pi^*}[\theta]\Big), \tag{39}$$

$$w_a'(\pi^*) = 2\,\frac{\partial}{\partial\pi}\Big(\mu_a - \mathbf{E}_{\pi}[\theta]\Big)\Big|_{\pi=\pi^*} = 2(\ell - h). \tag{40}$$

Given (38), and the previous two equations, we have

$$\frac{1}{2}\left(\frac{1-\lambda}{\pi^*} - \frac{1+\lambda}{1-\pi^*}\right) = \frac{w_a'(\pi^*)}{w_a(\pi^*)} = \frac{\ell - h}{\mu_a - \mathbf{E}_{\pi^*}[\theta]}. \tag{41}$$

Substituting $\mathbf{E}_{\pi^*}[\theta] = \pi^* h + (1-\pi^*)\ell$ and then solving (41) with respect to π^* implies $\pi^* = \frac{\mu_a - \ell + (\ell-\mu_a)\lambda}{2\mu_a - (\ell+h) + (\ell-h)\lambda}$. Comparing to the solution of the first best problem, we conclude $\pi^* = \pi_{\text{fb}}$, finishing the proof of (ii).

Moreover, we obtain from (39) that $\zeta_1 = 2\frac{2(h-\mu_a)}{\lambda-1}\left(\frac{\pi^*}{1-\pi^*}\right)^{\frac{1}{2}(1+\lambda)}$. As a result, $w_a''(\pi) = 2W_{avg}''(\pi)$ for $\pi > \pi^*$, proving the convexity of w_a.

To simplify the notation, we denote $g(\pi) = g(\pi, h, \ell)$ in the rest of the proof. It follows from (37) and (38) that $w_a'(\pi) = \left(\frac{1-\lambda}{2\pi} - \frac{1+\lambda}{2(1-\pi)}\right)w_a(\pi)$ and $w_a''(\pi) = \frac{2\sigma^2\rho}{2g(\pi)}w_a(\pi)$. Combining (7) and (30), we obtain that v satisfies

$$\begin{aligned} v(\pi) = \mu_a + \frac{g(\pi)}{2\rho\sigma^2}\Big(v''(\pi) - w_a''(\pi)\Big) &= \mu_a + \frac{g(\pi)}{2\rho\sigma^2}v''(\pi) - \frac{w_a(\pi)}{2} \\ &= \mu_a + \frac{g(\pi)}{2\rho\sigma^2}v''(\pi) - \frac{\zeta_1}{2}\pi^{\frac{1-\lambda}{2}}(1-\pi)^{\frac{1+\lambda}{2}} \end{aligned} \tag{42}$$

with initial condition $v(\pi^*) = \mathbf{E}_{\pi^*}[\theta] = h\pi^* + l(1-\pi^*)$ due to the value matching at π^*.

Next, we identify a unique bounded function v satisfying the previous differential equation and the initial condition. To this end, observe that the function $f(\pi) = \pi^{\frac{1-\lambda}{2}}(1-\pi)^{\frac{1+\lambda}{2}}$ obeys the differential equation $f''(\pi) = \frac{\lambda^2-1}{4}f(\pi) = \frac{\rho\sigma^2}{(h-\ell)^2}f(\pi)$. Therefore, define the function $\tilde{v}(\pi) = v(\pi) - \mu_a - A_1 f(\pi)$ for some constant A_1. Plug $\tilde{v}$ into (42) and using the previous equation, we obtain $A_1 = \frac{1}{2}A_1 - \frac{\zeta_1}{2}$ so that

$$A_1 = -\zeta_1 = -\frac{4(h-\mu_a)}{\lambda - 1}\left(\frac{\pi^*}{1-\pi^*}\right)^{\frac{1+\lambda}{2}},$$

and moreover, $\tilde{v}$ satisfies the equation $\tilde{v}(\pi) = \frac{g(\pi)}{2\rho\sigma^2}\tilde{v}''(\pi)$. Denote $\lambda_1 = \sqrt{1 + 8\frac{\sigma^2\rho}{(h-\ell)^2}}$. Then, the solution space for the previous differential equation is parametrized by two constants A_2 and C_2, $\tilde{v}(\pi) = A_2\pi^{\frac{1-\lambda_1}{2}}(1-\pi)^{\frac{1+\lambda_1}{2}} + C_2\pi^{\frac{1+\lambda_1}{2}}(1-\pi)^{\frac{1-\lambda_1}{2}}$. The boundedness of $\tilde{v}$ and λ_1 implies $C_2 = 0$.

Putting everything together, and using $w_a(\pi) = \zeta_1 f(\pi)$, v can be expressed as

$$v(\pi) = \mu_a - w_a(\pi) + A_2\pi^{\frac{1-\lambda_1}{2}}(1-\pi)^{\frac{1+\lambda_1}{2}} \tag{43}$$

where A_2 is chosen to satisfy the initial condition $v(\pi^*) = \pi^*h + (1-\pi^*)l$. Recall that $w_a(\pi^*) = 2(\mu_a - (\pi^*h + (1-\pi^*)l) = 2(\mu_a - v(\pi^*))$. Therefore, $v(\pi^*) = \mu_a - 2\mu_a + 2v(\pi^*) + A_2(\pi^*)^{\frac{1-\lambda_1}{2}}(1-\pi^*)^{\frac{1+\lambda_1}{2}}$ implies $A_2 = \frac{(\mu_a - v(\pi^*))}{(\pi^*)^{\frac{1-\lambda_1}{2}}(1-\pi^*)^{\frac{1+\lambda_1}{2}}}$. From the closed-form solution of v, we immediately get (iii), and we see that v is concave and that $\lim_{\pi\to 1} v(\pi) = \mu_a$.

Finally, we solve for w_b from (44). To this end, we obtain from (43) that $\frac{g(\pi)}{2\rho\sigma^2}(v''(\pi) + w_a''(\pi)) = A_2\pi^{\frac{1-\lambda_1}{2}}(1-\pi)^{\frac{1+\lambda_1}{2}}$. Plugging the previous expression into (44) and defining

$$\tilde{w}_b(\pi) = w_b(\pi) - 2(\pi h + (1-\pi)l - \mu_a) + 2A_2\pi^{\frac{1-\lambda_1}{2}}(1-\pi)^{\frac{1+\lambda_1}{2}},$$

we obtain that $\tilde{w}_b$ satisfies the equation $\tilde{w}_b(\pi) = \frac{g(\pi)}{\rho\sigma^2}\tilde{w}_b''(\pi)$. Since $\tilde{w}_b$ is also bounded, $\tilde{w}_b = 2B_2\pi^{\frac{1-\lambda}{2}}(1-\pi)^{\frac{1+\lambda}{2}}$ for some constant B_2. Therefore,

$$w_b(\pi) = 2(\pi h + (1-\pi)l - \mu_a) - 2A_2\pi^{\frac{1-\lambda_1}{2}}(1-\pi)^{\frac{1+\lambda_1}{2}} + 2B_2\pi^{\frac{1-\lambda}{2}}(1-\pi)^{\frac{1+\lambda}{2}}.$$

To determine B_2, we can use the initial condition $w_b(\pi^*) = 0$. Thus, $B_2 = \frac{2(\mu_a - v(\pi^*))}{(\pi^*)^{\frac{1-\lambda}{2}}(1-\pi^*)^{\frac{1+\lambda}{2}}}$. This closed-form solution shows that w_b is convex.

Finally, when $\pi = \pi^*$, (42) implies

$$v(\pi^*) = \mu_a + \frac{g(\pi^*)}{2\rho\sigma^2}v''(\pi^*) - \frac{w_a\pi^*}{2} = \mathbf{E}_{\pi^*}[\theta] + \frac{g(\pi^*)}{2\rho\sigma^2}v''(\pi^*),$$

where the second equation follows from (39). Due to the value matching of v at π^*: $v(\pi^*) = \mathbf{E}_{\pi^*}[\theta]$, we obtain $v''(\pi^*) = 0$, i.e., there is no information externality for buyers claimed in (18). Now combining (17) and (18), we confirm the conjecture (16). □

Proof of Lemma 3. From (38), we have

$$\begin{aligned} w_a'(\pi) =& \Big[\frac{1}{2}(1-\lambda)\pi^{-1} - \frac{1}{2}(1+\lambda)(1-\pi)^{-1}\Big]w_a(\pi), \\ w_a''(\pi) =& \frac{2\rho\sigma^2}{2g(\pi,h,\ell)}w_a(\pi). \end{aligned}$$

Because $\lambda > 1$ and w_a is positive, we see from the previous two equations that w_a' decreases and w_a'' increases when we increase the positive constant ζ_1 in (38).

Meanwhile, combining (10) and (30), w_b satisfies the equation

$$w_b(\pi) = 2\Big(\mathbf{E}_\pi[\theta] - \mu_a\Big) + \frac{1}{\rho\sigma^2}g(\pi,h,\ell)\Big(v''(\pi) + w_a''(\pi) + w_b''(\pi)\Big). \tag{44}$$

The comparison result for viscosity solutions implies that w_b increases with w_a''.

The previous two paragraphs combined implied that w_a and w_b are maximized when w_a'' is also maximized, which is equivalent to minimizing w_a'. However, for the maximization problem of w_a in (10), the notion of viscosity solution prohibits a concave kink. (see, e.g., Proposition 1 in Online Appendix D of Achdou et al. (2017)). Therefore, w_a' is minimal when w_a satisfies the smooth pasting condition we imposed at π^*. In this case, both profits for seller a and b are maximized.

□

Remark 2. *The smooth pasting for w_a can be interpreted in the following way. The seller a controls ξ_{ia} by choosing the price p_a that satisfies the conditions of Lemma 2. Then, seller a's value satisfies* *the HJB equation*

$$w_a(\pi) = \sup_{\xi_{ia}\in\{0,1\}} \Big\{ 2p_a(\pi)1_{\{\xi_{ia}=1\}} + g\frac{1}{\rho\sigma^2}w_a''(\pi)1_{\{\xi_{ib}=1\}} \Big\}.$$

This equation is equivalent to

$$0 = \max\Big\{ 2p_a(\pi) - w_a(\pi), \frac{1}{\sigma^2}gw_a''(\pi) - \rho w_a(\pi) \Big\},$$

which is exactly the variational inequality for the optimal stopping problem

$$w_a(\pi) = \sup_{\tau} \mathbf{E}_\pi\Big[e^{-\rho\tau}2p_a(\pi_\tau) \Big], \tag{45}$$

where τ is a stopping time chosen by seller a. We can interpret the value of this optimal stopping problem as the best value for the seller a when he can control the failure time τ of the product b. We anticipate that w_a is convex. Then, the optimal stopping time is a threshold type where the value matching and the smooth pasting conditions pin down the threshold.

Proof of Theorem 2. We often omit the argument for functions g, v_1, v_2, w_a, and w_b to simplify notation.

First, observe that by Proposition 2, the claim holds trivially unless the equilibrium is a cutoff one with $\pi_1^* > \pi_2^*$, under the usual interpretation that buyer i buys from seller b if and only if $\pi \geq \pi_i^*$. We are going to show that $\pi_1^* \neq \pi_{\text{fb},1}$, while $\pi_2^* = \pi_{\text{fb},2}$.

The proof has three main parts. We first simplify and relate the value functions of the market participants using the optimal pricing and purchasing strategies, obtaining a system of four nonlinear equations for the three regions, the one where none buys from b, the one where only buyer 2 buys from b, and the one where both buyers buy from b. This step also proves that the value of information for buyer 1 at the low cutoff is zero, i.e., $v_1''(\pi_2^*) = 0$.

Next, we combine the derived equations with value matching, smooth pasting, and supercontact conditions to numerically solve for the value function. The solution shows that at the high cutoff, there is a nonnull value of information for buyer 2, $v_2''(\pi_1^*) \neq 0$.

Finally, we conclude by showing that the good learner, who does not transit from product a to b at the high cutoff, is not incorporated in the price at the high cutoff. Therefore, the high cutoff is not the same as the efficient one.

Let us first write down HJB equations for v_1, v_2, w_a, w_b on $(1, \pi_2^*)$, (π_2^*, π_1^*), and $(\pi_1^*, 1)$.

Following (7), the HJB equation for buyer i is

$$v_i = \max\Big\{\mu_a - p_{a,i} + \frac{1}{2\rho} g \frac{\xi_{-ib}}{\sigma_{-i}^2} v_i'', \mathbf{E}_\pi[\theta] - p_{b,i} + \frac{1}{2\rho} g\Big(\frac{\xi_{-ib}}{\sigma_{-i}^2} + \frac{1}{\sigma_i^2}\Big) v_i'', \frac{g\xi_{-ib}}{2\rho\sigma_{-i}^2} v_i''\Big\}. \tag{46}$$

Following (9), the HJB equation for seller a is

$$w_a(\pi) = \sup_{p_a} \Big\{p_{a,1} 1_{\{\xi_{1a}(\pi, p_a, p_b(\pi))=1\}} + p_{a,2} 1_{\{\xi_{2a}(\pi, p_a, p_b(\pi))=1\}} + \frac{1}{2\rho} g\Big[\frac{1_{\{\xi_{1b}(\pi, p_a, p_b(\pi))=1\}}}{\sigma_1^2} + \frac{1_{\{\xi_{2b}(\pi, p_a, p_b(\pi))=1\}}}{\sigma_2^2}\Big] w_a''(\pi)\Big\}. \tag{47}$$

The HJB equation for seller b is

$$w_b(\pi) = \sup_{p_b} \Big\{p_{b,1} 1_{\{\xi_{1b}(\pi, p_a(\pi), p_b)=1\}} + p_{b,2} 1_{\{\xi_{2b}(\pi, p_a(\pi), p_b)=1\}} + \frac{1}{2\rho} g\Big[\frac{1_{\{\xi_{1b}(\pi, p_a(\pi), p_b)=1\}}}{\sigma_1^2} + \frac{1_{\{\xi_{2b}(\pi, p_a(\pi), p_b)=1\}}}{\sigma_2^2}\Big] w_b''(\pi)\Big\}. \tag{48}$$

The indifference condition for buyer i implies

$$p_{b,i} - p_{a,i} = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho} g \frac{1}{\sigma_i^2} v_i'', \quad i = 1, 2. \tag{49}$$

When b sells to buyer i, $p_{a,i} \leq \frac{1}{2\rho\sigma_i^2} g w_a''$ so that it is optimal for a not to sell to buyer i. Following our seller profit maximization pricing rule and the indifference condition (49), we set

$$p_{a,i} = \frac{1}{2\rho\sigma_i^2} g w_a'', \quad p_{b,i} = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_i^2} g\Big[v_i'' + w_a''\Big]. \tag{50}$$

When a sells to buyer i, $p_{b,i} \leq -\frac{1}{2\rho\sigma_i^2} g w_b''$ so that it is optimal for b not to sell to buyer i. The same argument as the previous case yields

$$p_{a,i} = \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_i^2} g\Big[v_i'' + w_b''\Big], \quad p_{b,i} = -\frac{1}{2\rho\sigma_i^2} g w_b''. \tag{51}$$

Let us specialize (46), (47), and (48) in three regions.

1. $\underline{\pi < \pi_2^*}$: Both buyers purchase a.

v_2 satisfies

$$v_2 = \max\left\{\mu_a - p_{a,2}, \mathbf{E}_\pi[\theta] - p_{b,2} + \frac{1}{2\rho\sigma_2^2} g v_2'', 0\right\} = \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_b''), \tag{52}$$

where the second equality follows from (51).

v_1 satisfies

$$v_1 = \max\left\{\mu_a - p_{a,1}, \mathbf{E}_\pi[\theta] - p_{b,1} + \frac{1}{2\rho\sigma_1^2} g v_1'', 0\right\} = \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b''), \tag{53}$$

where the second equality follows from (51).

w_a satisfies

$$w_a = p_{a,1} + p_{a,2} = 2\Big(\mu_a - \mathbf{E}_\pi[\theta]\Big) - \frac{1}{2\rho} g\Big[\frac{1}{\sigma_1^2} v_1'' + \frac{1}{\sigma_2^2} v_2'' + \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big) w_b''\Big], \tag{54}$$

where the second equality follows from (51).

w_b satisfies

$$w_b = 0. \tag{55}$$

We conjecture that $v_1'', v_2'', w_a'', w_b''$ are all zero in this region. Then

$$v_1 = \mathbf{E}_\pi[\theta], \quad v_2 = \mathbf{E}_\pi[\theta], \quad w_a = 2\Big(\mu_a - \mathbf{E}_\pi[\theta]\Big), \quad w_b = 0. \tag{56}$$

2. $\underline{\pi \in (\pi_2^*, \pi_1^*)}$: Buyer 1 purchases a, buyer 2 purchases b.

v_2 satisfies

$$v_2 = \max\left\{\mu_a - p_{a,2}, \mathbf{E}_\pi[\theta] - p_{b,2} + \frac{1}{2\rho\sigma_2^2} g v_2'', 0\right\} = \mu_a - \frac{1}{2\rho\sigma_2^2} g w_a'', \tag{57}$$

where the second equality uses (50).

v_1 satisfies

$$
\begin{aligned}
v_1 =& \max\Big\{\mu_a - p_{a,1} + g\frac{1}{2\rho\sigma_2^2}v_1'', \mathbf{E}_\pi[\theta] - p_{b,1} + \frac{1}{2\rho}g\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)v_1'', \frac{g}{2\rho\sigma_2^2}v_1''\Big\} \\
=& \mathbf{E}_\pi[\theta] + \frac{1}{2\rho}g\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)v_1'' + \frac{1}{2\rho\sigma_1^2}gw_b'',
\end{aligned}
\tag{58}
$$

where the second equality follows from (51).

w_a satisfies

$$
w_a = p_{a,1} + \frac{1}{2\rho}g\frac{1}{\sigma_2^2}w_a'' = \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2}g(v_1'' + w_b'') + \frac{1}{2\rho\sigma_2^2}gw_a'',
\tag{59}
$$

where the second equality follows from (51).

w_b satisfies

$$
w_b = p_{b,2} + \frac{1}{2\rho}g\frac{1}{\sigma_2^2}w_b'' = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2}g\Big(v_2'' + w_a'' + w_b''\Big),
\tag{60}
$$

where the second equality follows from (50).

3. $\underline{\pi > \pi_1^*}$: Both buyers purchase b.

v_2 satisfies

$$
v_2 = \max\Big\{\mu_a - p_{a,2} + \frac{gv_2''}{2\rho\sigma_1^2}, \mathbf{E}_\pi[\theta] - p_{b,2} + \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)\frac{gv_2''}{2\rho}, \frac{gv_2''}{2\rho\sigma_1^2}\Big\} = \mu_a - \frac{gw_a''}{2\rho\sigma_2^2} + \frac{gv_2''}{2\rho\sigma_1^2},
\tag{61}
$$

where the second equality uses (50).

v_1 satisfies

$$
v_1 = \max\Big\{\mu_a - p_{a,1} + \frac{gv_1''}{2\rho\sigma_2^2}, \mathbf{E}_\pi[\theta] - p_{b,1} + \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)\frac{gv_1''}{2\rho}, \frac{gv_1''}{2\rho\sigma_2^2}\Big\} = \mu_a - \frac{gw_a''}{2\rho\sigma_1^2} + \frac{gv_1''}{2\rho\sigma_2^2},
\tag{62}
$$

where the second equality follows from (50).

w_a satisfies

$$
w_a = \frac{1}{2\rho}g\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)w_a''.
\tag{63}
$$

w_b satisfies

$$w_b = p_{b,1} + p_{b,2} + \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)\frac{g w_b''}{2\rho} = 2\Big(\mathbf{E}_\pi[\theta] - \mu_a\Big) + \frac{g}{2\rho}\Big[\frac{v_1''}{\sigma_1^2} + \frac{v_2''}{\sigma_2^2} + \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)\Big(w_a'' + w_b''\Big)\Big], \tag{64}$$

where the second equality follows from (50).

After writing down HJB equations on different regions, we claim that these equations are equivalent to the following four HJB equations:

$$w_a = \max\Big\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{g(v_2'' + w_b'')}{2\rho\sigma_2^2}, \frac{g w_a''}{2\rho\sigma_2^2}\Big\} + \max\Big\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{g(v_1'' + w_b'')}{2\rho\sigma_1^2}, \frac{g w_a''}{2\rho\sigma_1^2}\Big\}. \tag{65}$$

$$\begin{aligned} w_b = &\max\Big\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_a'') + \frac{1}{2\rho\sigma_2^2} g w_b'', 0\Big\} \\ &+ \max\Big\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_a'') + \frac{1}{2\rho\sigma_1^2} g w_b'', 0\Big\}. \end{aligned} \tag{66}$$

$$\begin{aligned} v_1 = &\min\Big\{\mu_a - \frac{1}{2\rho\sigma_1^2} g w_a'', \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b'')\Big\} \\ &+ \frac{1}{2\rho\sigma_2^2} g v_1'' 1_{\{\text{first term in the } v_2 \text{ equation is smaller}\}} \end{aligned} \tag{67}$$

$$\begin{aligned} v_2 = &\min\Big\{\mu_a - \frac{1}{2\rho\sigma_2^2} g w_a'', \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_b'')\Big\} \\ &+ \frac{1}{2\rho\sigma_1^2} g v_2'' 1_{\{\text{first term in the } v_1 \text{ equation is smaller}\}}. \end{aligned} \tag{68}$$

To see how equations (65) - (68) are equivalent to HJB equations of v_1, v_2, w_a, and w_b on different regions, we first consider the case where the LHS of (23), (24) < 0. In this case, first terms in both maximization of (65) are larger than the second term in the same maximization problem. As a result, (65) agrees with (54). Meanwhile, the first term in both maximization of (66) are smaller than the second term, so that (66) agrees with (55). Both first term in the minimization of (67) and (68) are smaller than their corresponding second term. Therefore equations (67) and (68) agree with (53) and (52), respectively. Therefore, equations (65) - (68) agree with the HJB equations of v_1, v_2, w_a, w_b on $(0, \pi_2^*)$. Using the similar argument, we can show these equations also agree on (π_2^*, π_1^*) and $(\pi_1^*, 1)$.

In what follows, we derive two consequences of (23) and (24).

1. When $\pi \in (\pi_2^*, \pi_1^*)$, we obtain from adding up (57) to (60) that

$$\begin{aligned} v_1 + v_2 + w_a + w_b =& \mathbf{E}_\pi[\theta] + \mu_a + \frac{1}{2\rho\sigma_2^2} g[v_1'' + v_2'' + w_a'' + w_b''] \\ =& 2\mu_a + \frac{1}{2\rho\sigma_2^2} g v_1'' + \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g[v_2'' + w_a'' + w_b'']. \end{aligned}$$

   Evaluate the previous equation at π_2^*. If $v_1 + v_2 + w_a + w_b$ is continuous at π_2^*, the left hand side is $2\mu_a$. The right-hand side, equation (23) implies that

$$v_1''(\pi_2^*) = 0. \tag{69}$$

2. When $\pi \in (\pi_2^*, \pi_1^*)$, it follows from (59) that

$$\begin{aligned} w_a =& \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b'') + \frac{1}{2\rho\sigma_2^2} g w_a'' \\ =& \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_a'' + w_b'') + \frac{1}{2\rho}\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big) g w_a''. \end{aligned}$$

   Evaluate the previous equation at π_1^*, use (24) and the equation of w_a when $\pi > \pi_1^*$ (see (59)). We obtain that when w_a is continuous at π_1^*, w_a'' is also continuous at π_1^*.

Solving the system of equations (65) - (68) is numerically challenging. However, we can solve the HJB equations for v_1, v_2, w_a, and w_b on three regions and combine the solutions with appropriate value matching, smooth pasting, and super-contact conditions. The two observations above motivate us to impose the following value matching, smooth pasting, and super contact conditions: (1) Value matching for all the value functions at π_1 and π_2. (2) Smooth pasting of w_a at π_1 and π_2. (3) Super-contact of w_a at π_1, i.e., $\lim_{\pi\uparrow\pi_1} w_a''(\pi) = \lim_{\pi\downarrow\pi_1} w_a''(\pi)$. (4) Smooth pasting of v_2 at π_1. (5) Smooth pasting of v_1 at π_2. (6) Smooth pasting of w_b at π_2. Our numeric solutions are presented in Figures 3 and 4. The proofs for $\pi_2^* = \pi_{\text{fb},2}$ and $\pi_1^* \neq \pi_{\text{fb},1}$ are already presented in the main text. □

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Online Appendix: Omitted proofs

Proof of Lemma 1. We focus on the time period where at least one buyer consumes the product b, i.e., $\sum_i \xi_{bi}(t) > 0$. On the complement of this set, the statement of the Lemma holds trivially, because $\xi_{b1}(t) = \xi_{b2}(t) = 0$, hence no additional information arrives and $d\pi_t = 0$.

Define $\vartheta = \frac{\theta-\ell}{h-\ell}$. Then we have $\vartheta \in \{0,1\}$, $\theta = h\vartheta + \ell(1-\vartheta)$, and $\pi_t = \mathbf{E}[\vartheta|\mathcal{F}_t]$. When $\xi_{ib}(t) = 1$ for $i = 1$ or 2, $C_{bi}(t)$ is observable and it follows the dynamics

$$dC_{bi}(t) = \Big(h\vartheta + \ell(1-\vartheta)\Big)dt + \sigma_i dZ_{it}.$$

It then follows from Liptser and Shiryaev (2001) Theorem 9.1 that the innovation process

$$d\widetilde{Z}_{it} = \frac{1}{\sigma_i}\Big[dC_{bi}(t) - \Big(h\pi_t + \ell(1-\pi_t)\Big)dt\Big]$$

is a $\mathcal{F}$-Brownian motion on the set $\{t \geq 0 : \xi_{ib}(t) = 1\}$ and $\widetilde{Z}_{1t}$ and $\widetilde{Z}_{2t}$ are independent. Moreover, π_t satisfies the filtering equation

$$d\pi_t = (h-\ell)\Big(\mathbf{E}[\vartheta^2|\mathcal{F}_t] - \mathbf{E}[\vartheta|\mathcal{F}_t]^2\Big)\sum_{i=1}^{2}\frac{\xi_{bi}(t)}{\sigma_i}d\widetilde{Z}_{it}.$$

Because ϑ is either 0 or 1, $\mathbf{E}[\vartheta^2|\mathcal{F}_t] - \mathbf{E}[\vartheta|\mathcal{F}_t]^2 = \mathbf{E}[\vartheta|\mathcal{F}_t] - \mathbf{E}[\vartheta|\mathcal{F}_t]^2 = \pi_t(1-\pi_t)$. Meanwhile, when $\sum_{i=1}^{2}\xi_{bi}(t) > 0$, since $\xi_{bi}(t) \in \{0,1\}$ for all $i \in \{1,2\}$ we can define an 1-dimensional $\mathcal{F}$-Brownian motion Z via

$$dZ_t = \Big(\sum_{j=1}^{2}\frac{\xi_{bj}(t)}{\sigma_j^2}\Big)^{-\frac{1}{2}}\sum_{i=1}^{2}\frac{\xi_{bi}(t)}{\sigma_i}d\widetilde{Z}_{it}.$$

Combining the previous two equations, we obtain

$$d\pi_t = \pi_t(1-\pi_t)(h-\ell)\sqrt{\sum_{i=1,2}\frac{\xi_{bi}(t)}{\sigma_i^2}}dZ_t.$$

$\square$

Proof of Proposition 8. First, condition (iii), $\sigma_1 > \sigma_2$, and the convexity of W_{avg} imply that

$$
\begin{aligned}
&0 > \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)W''_{avg}(\pi)}{\rho\sigma_2^2} \geq \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{\rho\sigma_1^2}W''_{avg}(\pi), && \forall \pi \in (0, \pi_{\text{fb},2}),\\
&\mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{\rho\sigma_2^2}W''_{avg}(\pi) > 0 > \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{\rho\sigma_1^2}W''_{avg}(\pi), && \forall \pi \in (\pi_{\text{fb},2}, \pi_{\text{fb},1}),\\
&\mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{\rho\sigma_2^2}W''_{avg}(\pi) \geq \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{\rho\sigma_1^2}W''_{avg}(\pi) > 0, && \forall \pi \in (\pi_{\text{fb},1}, 1).
\end{aligned}
$$

Therefore, combining the previous inequalities with the condition (i), we obtain that W_{avg} satisfies (5) and the optimizers $\xi^*_{ib}(\pi) = 1_{(\pi_{\text{fb},i},\infty)}(\pi)$, $i = 1,2$.

Next, we verify W_{avg} is the first best value function and ξ^*_{ib} in (6) is a first best optimal strategy. To this end, because $W_{avg} \in C^2(\pi_{\text{fb},i+1}, \pi_{\text{fb},i})$, $i \in \{0,1,2\}$, and W_{avg} satisfies (27), Karatzas and Shreve (1998) Chapter 3 Problem 6.24 implies that Itô's formula can be applied to this type of piecewise C^2 function. Consider any pair of strategies $\{\xi_{ib}\}_{i=1,2}$. Itô's formula implies that

$$
\begin{aligned}
&d\Big\{e^{-\rho t}W_{avg}(\pi_t) + \int_0^t \rho e^{-\rho s}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\Big(\mathbf{E}_{\pi_s}[\theta] - \mu_a\Big)\Big)ds\Big\}\\
&= \rho e^{-\rho t}\Big\{-W_{avg}(\pi_t) + \frac{1}{2\rho}g(\pi,h,\ell)W''_{avg}(\pi_t)\sum_{i=1}^{2}\frac{\xi_{ib}(t)}{\sigma_i^2} + \Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(t)\Big(\mathbf{E}_{\pi_t}[\theta] - \mu_a\Big)\Big)\Big\}dt\\
&\quad + \text{martingale},
\end{aligned}
$$

whose drift is nonpositive due to (5). Therefore, the process

$$
e^{-\rho t}W_{avg}(\pi_t) + \int_0^t \rho e^{-\rho s}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\Big(\mathbf{E}_{\pi_s}[\theta] - \mu_a\Big)\Big)ds
$$

is a local supermartingale. Because both W_{avg} and $\mu_a + \frac{1}{2}\sum_{i=1}^2 \xi_{ib}(t)\Big(\mathbf{E}_{\pi_t}[\theta] - \mu_a\Big)$ are bounded, by Lemma 5.6.8 in Cohen and Elliott (2015), the previous process is a supermartingale as well. As a result, for any T,

$$
W_{avg}(\pi_t) \geq \mathbf{E}_t\Big[e^{-\rho(T-t)}W_{avg}(\pi_T) + \int_t^T \rho e^{-\rho(s-t)}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\Big(\mathbf{E}_{\pi_s}[\theta] - \mu_a\Big)\Big)\,ds\Big].
$$

Sending $T \to \infty$, because W_{avg} is bounded, it satisfies the transversality condition

$$\lim_{T\to\infty} \mathbf{E}_t[e^{-\rho(T-t)} W_{avg}(\pi_T)] = 0.$$

The previous two expressions combined yield

$$W_{avg}(\pi_t) \geq \mathbf{E}_t\Big[\int_t^{\infty} \rho e^{-\rho(s-t)}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2} \xi_{ib}(s)\Big(\mathbf{E}_{\pi_s}[\theta] - \mu_a\Big)\Big)\, ds\Big],.$$

for any strategy ξ_{ib}. When the strategy is chosen as in (6), the inequality above is an equality, confirming the optimality of ξ^*_{ib}. □

Proof of Claim 1. Sending π approaching $\pi_{\text{fb},1}$ from the left and right, $W_{avg,0}$ and $W_{avg,1}$ satisfy

$$W_{avg,1}(\pi_{\text{fb},1}) = \mu_a + \frac{1}{2}\Big(\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a\Big) + \frac{1}{2\rho\sigma_2^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,1}(\pi_{\text{fb},1}) \tag{70}$$

$$\begin{aligned} W_{avg,0}(\pi_{\text{fb},1}) =& \mu_a + \frac{1}{2}\Big(\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a\Big) + \frac{1}{2\rho\sigma_2^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,0}(\pi_{\text{fb},1}) \\ &+ \frac{1}{2}\Big(\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a\Big) + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,0}(\pi_{\text{fb},1}). \end{aligned} \tag{71}$$

Suppose that $W''_{avg,1}(\pi_{\text{fb},1}) < W''_{avg,0}(\pi_{\text{fb},1})$. Then the right-hand side of (70) is less than the first line on the right-hand side of (71). However, since the left-hand sides of (70) and (71) have the same value due to the value matching, the second line on the right-hand side of (71) must be strictly less than 0. This contradicts with Proposition 8 (iii) that

$$\mathbf{E}_{\pi}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) W''_{avg,0}(\pi) > 0 \text{ for } \pi \text{ in a right neighborhood of } \pi_{\text{fb},1}.$$

Suppose that $W''_{avg,1}(\pi_{\text{fb},1}) > W''_{avg,0}(\pi_{\text{fb},1})$. Then the condition

$$\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,0}(\pi_{\text{fb},1}) \geq 0$$

in Proposition 8 (iii) implies

$$\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,1}(\pi_{\text{fb},1}) > 0$$

Due to the continuity of $W''_{avg,1}$ on $(\pi_{\text{fb},2}, \pi_{\text{fb},1})$, the previous inequality implies

$$\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) W''_{avg,1}(\pi) > 0 \text{ in a left neighborhood of } \pi_{\text{fb},1}.$$

However, this contradicts with the condition Proposition 8 (iii) on $(\pi_{\text{fb},2}, \pi_{\text{fb},1})$.

Combining the previous two cases, we verify the claim. □

Proof of Proposition 3. We start characterizing the equilibrium prices posted by the monopolist in a cutoff equilibrium when buyers are symmetric.

Claim 2. *The prices are as follows in every symmetric equilibrium with cutoff π_m^*. If $\pi \leq \pi_m^*$*

$$p_a(\pi) = \mu_a \qquad \textit{and} \qquad p_b(\pi) \geq \mathbf{E}_\pi[\theta]. \tag{72}$$

If $\pi > \pi_m^*$

$$p_a(\pi) \geq \mu_a \qquad \textit{and} \qquad p_b(\pi) = \mathbf{E}_\pi[\theta]. \tag{73}$$

To prove this result, as a preliminary observation, notice that the problem of the monopolist reduces to the choice between either selling the product of known quality or selling the one of unknown quality. Indeed, given this choice, there is no reason to charge less than the buyers' maximal willingness to pay since the informational content generated by the use of the product is unaffected by the price. Therefore, if $\pi < \pi_m^*$, we have $p_a(\pi) = \mu_a$. Since by the definition of cutoff equilibrium the monopolist is selling the product of known quality at those beliefs, it immediately follows from the value function of the buyer that $p_b(\pi) \geq \mathbf{E}_\pi[\theta] + \frac{g(\pi,h,\ell)}{2\rho\sigma_i^2} v''(\pi)$. Similarly, if $\pi \geq \pi_m^*$, the monopolist sets a price of product b equal to the willingness to pay $p_b(\pi) = \mathbf{E}_\pi[\theta] + \frac{g(\pi,h,\ell)}{2\rho\sigma_i^2} v''(\pi)$. It follows from the value function of the buyers that $p_a(\pi) \geq \mu_a$. However, if we plug these prices into the value function of the buyers, we obtain that v is equal to 0, and so is its second derivative v''. This, together with the previously computed prices, implies the result.

Next, how do we find the threshold π_m^*? Recall that, once π reaches π_m^* from above, π stops at π_m^*, product b fails, and only product a is offered from then on. Given the pricing strategy of the monopolist characterized in Claim 2, we know the value function of the monopolist for beliefs below the threshold. Therefore, to find the threshold, we combine a smooth pasting and a value-matching condition with the second-order ODE given by the diffusion process derived in Lemma 1. First, notice that the continuation value of each market participant is always nonnegative since they all have a strategy that guarantees a deterministic zero payoff. At the same time, observe that given the pricing strategies of

Claim 2 for some cutoff π_m, it is optimal for the buyers to use the strategies

$$\xi_{i,a}(\pi, p_a, p_b) = 1 \iff \mu_a - p_{a,i} = \max\{\mu_a - p_{a,i}, \mathbf{E}_\pi[\theta] - p_{b,i}, 0\} \text{ and } \pi \leq \pi_m \tag{74}$$

$$\xi_{i,b}(\pi, p_a, p_b) = 1 \iff \mathbf{E}_\pi[\theta] - p_{b,i} = \max\{\mu_a - p_{a,i}, \mathbf{E}_\pi[\theta] - p_{b,i}, 0\} \text{ and } \pi > \pi_m. \tag{75}$$

Moreover, the induced expected discounted utility for the buyers is equal to 0. Therefore, the monopolist obtains the first-best welfare by setting the cutoff equal to the welfare-maximizing one. Since we have noted that the continuation utilities of all market participants have to be nonnegative, using that cutoff is optimal for the monopolist. □

Proof of Proposition 4. (i) The proof follows the same lines of Lemma 2 and Proposition 3. In particular, consider the pricing strategy

$$p_a(\pi) = \mu_a \qquad \textit{and} \qquad p_b(\pi) = \mathbf{E}_\pi[\theta]. \tag{76}$$

It is immediate to see that under this pricing strategy, the buyers have a value function that is identically 0, and they are always indifferent between the two products. Therefore, by letting

$$\xi_{i,a}(\pi, p_a, p_b) = 1 \iff \mu_a - p_{a,i} = \max\{\mu_a - p_{a,i}, \mathbf{E}_\pi[\theta] - p_{b,i}, 0\} \text{ and } \pi \leq \pi_m \tag{77}$$

$$\xi_{i,b}(\pi, p_a, p_b) = 1 \iff \mathbf{E}_\pi[\theta] - p_{b,i} = \max\{\mu_a - p_{a,i}, \mathbf{E}_\pi[\theta] - p_{b,i}, 0\} \text{ and } \pi > \pi_m. \tag{78}$$

we obtain an equilibrium that is welfare-maximizing. (ii) The result where the monopolist is forced to use the same price for both products is trivial because by Proposition 2 the first-best features two different thresholds. □

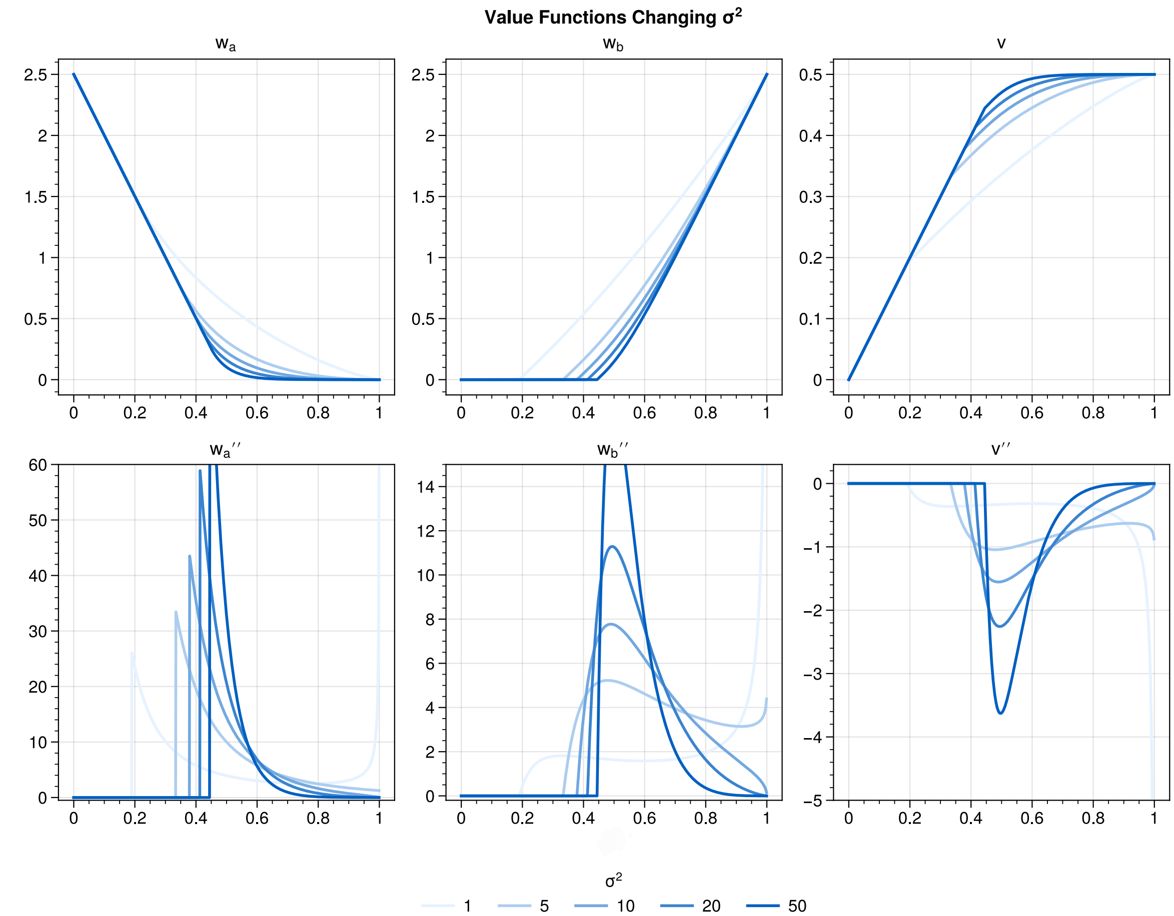

Figure 5: This figure plots the value functions w_a, w_b and v and their second derivatives w_a'', w_b'' and v'' when σ^2 changes, fixing other parameters to $h = 1, \ell = 0, \mu_a = .5$ (using the explicit characterizations of the value functions in the proof of Theorem 1). As shown in the figure, with increasing σ^2 the cutoff π^*, expectedly, moves to the right (i.e., it increases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

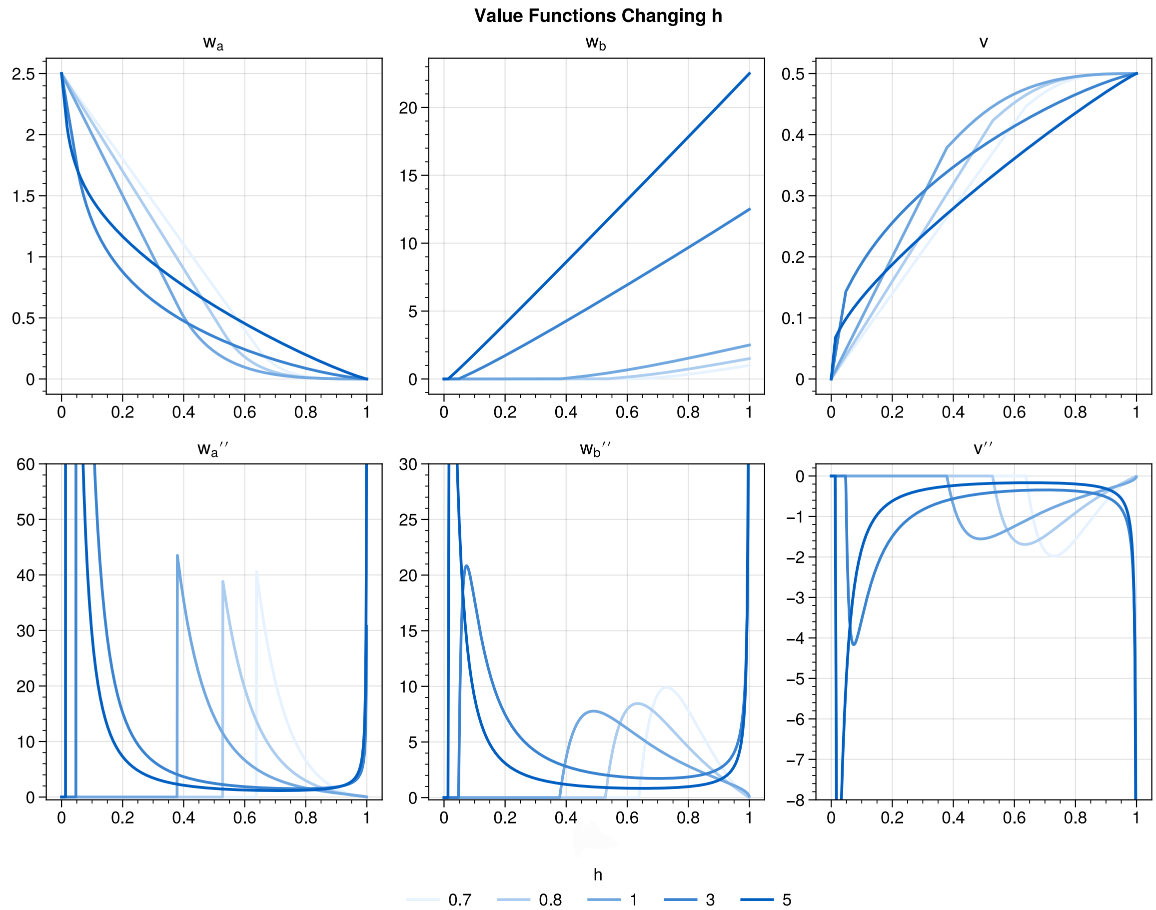

Figure 6: This figure plots the value functions w_a, w_b and v and their second derivatives w_a'', w_b'' and v'' when h changes, fixing other parameters to $\sigma^2 = 10$, $\ell = 0$, $\mu_a = .5$ (using the explicit characterizations of the value functions in the proof of Theorem 1). As shown in the figure, with increasing h the cutoff π^*, expectedly, moves to the left (i.e., it decreases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

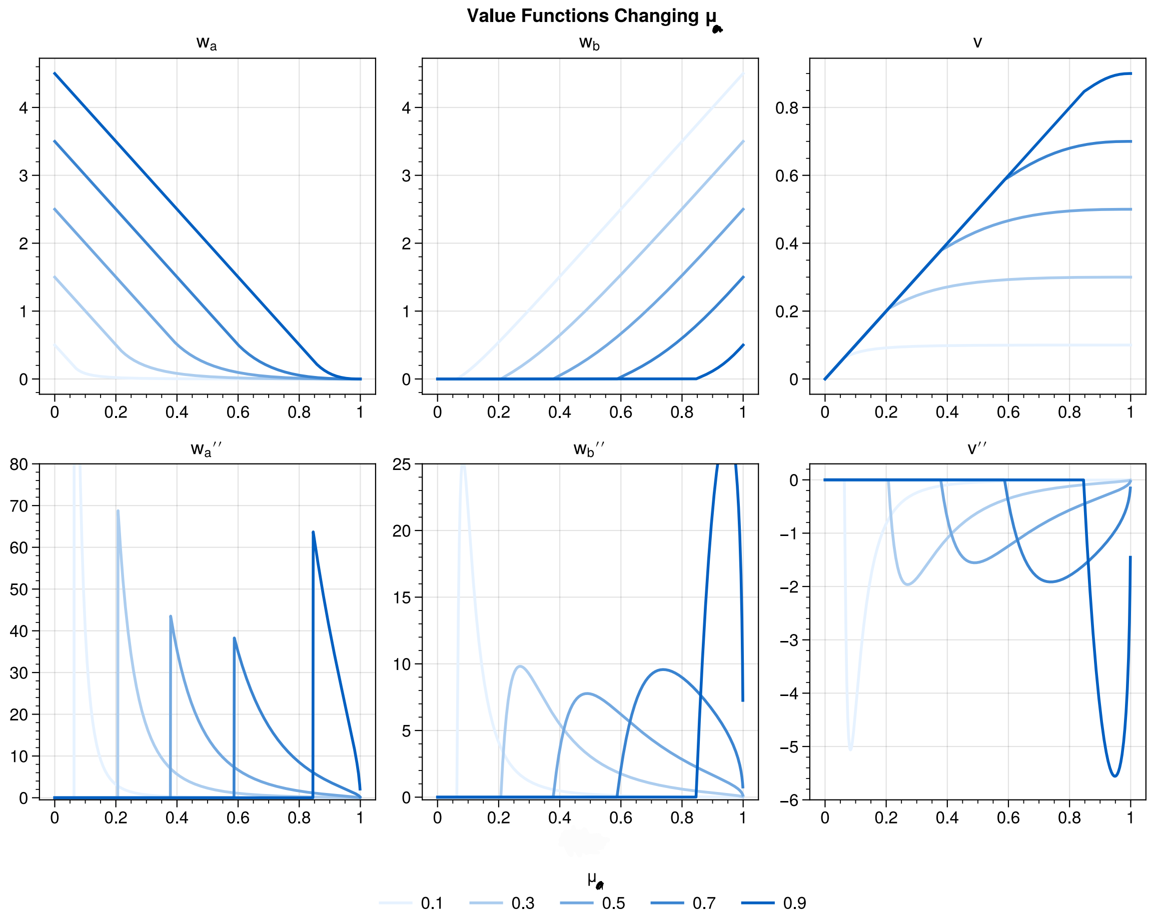

Figure 7: This figure plots the value functions w_a, w_b and v and their second derivatives w_a'', w_b'' and v'' when μ_a changes, fixing other parameters to $\sigma^2 = 10, \ell = 0, h = 1$ (using the explicit characterizations of the value functions in the proof of Theorem 1). As shown in the figure, with increasing μ_a the cutoff π^*, expectedly, moves to the right (i.e., it increases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

Proof of Proposition 5. The result follows immediately by rewriting equation (24) as

$$\left(\mu_a - \mathbf{E}_{\pi_1^*}[\theta]\right) 2\rho\sigma_1^2 = g\left(\pi_1^*, h, \ell\right)\left(v_1''\left(\pi_1^*\right) + w_b''\left(\pi_1^*\right) + w_a''\left(\pi_1^*\right)\right). \tag{79}$$

Notice that as σ_1 goes to infinity the LHS goes to ∞ unless $\pi_1^* \to \pi_{myopic}$. Observe that the value functions of the agents are uniformly bounded by $2h$ across all the values of σ_1. Therefore, in a right neighborhood of π_1, we have that

1. w_a'' is bounded due to equation (63).

2. v_1'' is bounded due to equation (62) and the point 1 above.

3. v_2'' is bounded due to equation (61) and the point 1 above.

4. w_b'' is bounded due to equation (64) and the points 1,2 and 3 above,

proving that the RHS is not going to ∞ and $\pi_1^* \to \pi_{myopic}$. Since the same holds for the first best, we obtain the result. $\square$

Proof of Proposition 6. Let $p_{a,1}, p_{a,2}, p_{a,1}, p_{b,2}$ denote the equilibrium prices under competition without multilateral contracts (cf. Theorem 2). It is just bookeeping to check that the following profile of Markov strategies is a welfare maximizing equilibrium.

- Seller a continues to use the pricing strategies in the equilibrium without multilateral contracts:

$$
\begin{aligned}
t_{a,1}^1(\pi) &= p_{a,1}(\pi) \\
t_{a,2}^2(\pi) &= p_{a,2}(\pi) \\
t_{a,1}^2(\pi) &= -v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2} \\
t_{a,2}^1(\pi) &= -v_1''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_2^2}.
\end{aligned}
$$

- Seller b asks for the transfers:

$$
\begin{aligned}
t_{b,1}^1(\pi) &= p_{b,1}(\pi) \\
t_{b,2}^2(\pi) &= p_{b,2}(\pi) \\
t_{b,1}^2(\pi) &= v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2} \\
t_{b,2}^1(\pi) &= v_1''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_2^2}.
\end{aligned}
$$

- Buyers accept the multilateral contract $\left(t_{b,i}^1, t_{b,i}^2\right)$, $i \in \{1,2\}$ if $\pi \geq \pi_{\text{fb},i}$ and the multilateral contract $\left(t_{a,i}^1, t_{a,i}^2\right)$ otherwise.

Indeed, recall that under the first best satiation, welfare maximizing condition imposes

$$
\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2}g(\pi_{\text{fb},1})\left(w_a'' + w_b'' + v_1'' + v_2''\right)(\pi_{\text{fb},1}) = 0.
$$

Instead the equilibrium cutoff is determined by where both seller a and seller b are indifferent to sell to buyer 1. Seller a's indifference condition:

$$\begin{aligned} t^1_{a,1}(\pi^*_1) + t^2_{a,1}(\pi^*_1) =& \mu_a - \mathbf{E}_{\pi^*_1}[\theta] - \frac{1}{2\rho\sigma_1^2} g(\pi^*)\Big(v_1''(\pi^*_1) + w_b''(\pi^*_1)\Big) - \frac{1}{2\rho\sigma_1^2} g(\pi^*_1) v_2''(\pi^*_1) \\ =& -\frac{1}{2\rho\sigma_1^2} g(\pi^*_1) w_a''(\pi^*_1). \end{aligned}$$

Seller b's indifference condition:

$$\begin{aligned} t^1_{b,1}(\pi^*_1) + t^2_{b,1}(\pi^*_1) =& \mathbf{E}_{\pi^*_1}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi^*_1)\Big(v_1''(\pi^*_1) + w_a''(\pi^*_1)\Big) + \frac{1}{2\rho\sigma_1^2} g(\pi^*_1) v_2''(\pi^*_1) \\ =& -\frac{1}{2\rho\sigma_1^2} g(\pi^*_1) w_b''(\pi^*_1). \end{aligned}$$

Similar equalities hold at π^*_2 as well.

□

Proof of Proposition 7. We first note that the corresponding comparative statistics are immediate from the explicit characterizations of $\mathbf{E}_{\pi_0}[\tau]$, Pr_{π_0}\{full adoption\} and Pr_{π_0}\{discarding\} and the fact that the endogenous π_2 and π_1 do not depend on π_0 (see Proposition 2). Next, we explicitly derive $\mathbf{E}_{\pi_0}[\tau]$, Pr_{π_0}\{full adoption\} and Pr_{π_0}\{discarding\}. To prove this statement we prepare two preliminary results.

Lemma [Extended Feynman-Kac Formula]. *Let $\Phi(x), f(x), F(x)$, $x \in [\pi_2, \pi_1]$, be continuous functions (f is non-negative). Let $u(x)$, $x \in [\pi_2, \pi_1]$ be a solution to*

$$\frac{\sigma^2(x)}{2} u''(x) - (\lambda + f(x))u(x) = -\lambda\Phi(x) - F(x), \quad x \in [\pi_2, \pi_1]$$

and $u(\pi_2) = \Phi(\pi_2)$ and $u(\pi_1) = \Phi(\pi_1)$ then

$$u(x) = \mathbf{E}_x\left[\Phi\Big(\pi_{\tau\wedge\mathcal{H}(\pi_2,\pi_1)}\Big)e^{-\int_0^{\tau\wedge\mathcal{H}(\pi_2,\pi_1)}} f(\pi_s)ds + \int_0^{\tau\wedge\mathcal{H}(\pi_2,\pi_1)} F(\pi_s)e^{-\int_0^s f(\pi_r)dr} ds\right]$$

where τ is random variable with the density $\lambda e^{-\lambda t}\mathbf{1}_{t\in[0,\infty)}$.

The proof of this result follows by a simple extension of the celebrated Feynman-Kac formula, omitted. Next, for ease of notation let us define $\mathcal{H}(\pi_2, \pi_1) = \inf\{t : \pi_t \notin (\pi_2, \pi_1)\}$.

Lemma 4. $\mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)] < \infty$.

Proof. The proof follows from the extended Feynman-Kac Formula. To show it, consider a family of functions $\{u_\lambda(x) : x \in [\pi_2, \pi_1]\}_{\lambda \geq 0}$ that are solution to the following $\lambda-$parametric problem:

$$\frac{\sigma^2(x)}{2} u''(x) - \lambda u(x) = -1, \quad x \in [\pi_2, \pi_1] \tag{80}$$

and $u(\pi_2) = u(\pi_1) = 0$. From the extended Feynman-Kac Formula it follows that $u_\lambda(x) = \mathbf{E}_x [\tau \wedge \mathcal{H}(\pi_2, \pi_1)]$ for $\lambda > 0$. Next, we argue that $\sup_{\lambda > 0} u_\lambda(x) \leq u_0(x)$, where $u_0(x)$ solves (80) when $\lambda = 0$.

Next, since $\lim_{\lambda \to 0} \tau = \infty$ thus $\lim_{\lambda \to 0} \tau \wedge \mathcal{H}(\pi_2, \pi_1) = \mathcal{H}(\pi_2, \pi_1)$. Therefore $\mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)] < \infty$, finishing the proof. □

Next, we present two useful corollaries.

Corollary 1. *Let $f(x)$ and $F(x)$, $x \in [\pi_2, \pi_1]$, be continuous functions and $f(x)$ be non-negative. Let the function Φ be defined only at two points π_2 and π_1. Then the function*

$$q(x) = \mathbf{E}_x \left[\Phi(\pi_{\mathcal{H}(\pi_2,\pi_1)}) e^{-\int_0^{\mathcal{H}(\pi_2,\pi_1)} f(\pi_s) ds} + \int_0^{\mathcal{H}(\pi_2,\pi_1)} F(\pi_s) e^{-\int_0^s f(\pi_r) dr} ds \right] \tag{81}$$

is the solution of the following problem

$$\frac{\sigma^2(x)}{2} q''(x) - f(x) q(x) + F(x) = 0, \quad x \in [\pi_2, \pi_1], \tag{82}$$

and $q(\pi_2) = \Phi(\pi_2)$ and $q(\pi_1) = \Phi(\pi_1)$.

The proof of this corollary follows directly from the extended Feynman-Kac Formula by assuming $\lambda = 0$, replacing $u(x)$ with $q(x)$.

Corollary 2. *The solution to the problem*

$$\frac{\sigma^2(x)}{2} q''(x) + F(x) = 0, \quad x \in [\pi_2, \pi_1],$$

$q(\pi_2) = \Phi(\pi_2)$ and $q(\pi_1) = \Phi(\pi_1)$ has the following form

$$q(x) = \frac{\pi_1 - x}{\pi_{\mathrm{fb},1} - \pi_2} \left(\Phi(\pi_2) + \int_{\pi_2}^{x} (y - \pi_2) \frac{2F(y)}{\sigma^2(y)} dy \right) + \frac{x - \pi_2}{\pi_1 - \pi_2} \left(\Phi(\pi_1) + \int_x^{\pi_1} (\pi_1 - y) \frac{2F(y)}{\sigma^2(y)} dy \right).$$

The proof of this corollary directly follows from extended Feynman-Kac Formula.

Using the above two corollaries, we have $\Pr_{\pi_0}\{\pi_\tau = \pi_2\} = \frac{\pi_1-\pi_0}{\pi_1-\pi_2}$ and $\Pr_{\pi_0}\{\pi_\tau = \pi_1\} = \frac{\pi_0-\pi_2}{\pi_1-\pi_2}$. These results follow from the above corollaries by assuming $F = f = 0, \Phi(\pi_2) = 1$ and $\Phi(\pi_1) = 0$.

In addition

$$
\begin{aligned}
\mathbf{E}_{\pi_0}[\tau] &= \mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)] = \\
&= \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2} \int_{\pi_2}^{\pi_0} (y - \pi_2) \frac{2dy}{\sigma^2(y)} + \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2} \int_{\pi_0}^{\pi_1} (\pi_1 - y) \frac{2dy}{\sigma^2(y)}
\end{aligned}
$$

which follows by the above corollaries by assuming $F = 1, f = 0, \Phi(\pi_2) = \Phi(\pi_1) = 0$ (implying $q(\pi_0) = \mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)]$ is the solution to (82)).

By these results, the proof of the proposition is now complete.

□

Viscosity solution

This section shows that buyer i value function can be characterized as a viscosity solution to its associated dynamic programming equation. The same statement holds for sellers' value functions.

Define buyer i's value function for a given pricing strategy of the sellers as

$$
v_i(\pi) = \sup_{\xi_{ik}, k \in \{a,b\}} \mathbf{E}\left[\int_0^\infty \rho e^{-\rho t}\, \xi_{ik}(t)\Big(dC_{ki}(t) - p_{k,i}(t)dt\Big)\right]. \tag{83}
$$

Proposition 9. *If p is an equilibrium pricing strategy for the seller, then v_i is a viscosity solution to equation* (7)*.*

Proof. The result is standard in the theory of viscosity solutions. If v_i is locally bounded, the results follow from Propositions 4.3.1 and 4.3.2 in Pham (2009). To verify that v_i is locally bounded, we will show v_i is globally bounded on $[0, 1]$. To this end, choosing a sub-optimal strategy $\xi_{ik} \equiv 0$ guarantees that v_i is nonnegative. Moreover, the continuation value of each market participant is always weakly smaller than W, which is globally bounded from above by nh. Therefore, v_i is also globally bounded from above by nh.

□