On the Negative Performance Puzzle: Positive Alpha and Investor Value

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Abstract

We propose a parsimonious equilibrium model of active managers and investors. We allow feedback effects from optimal fee-setting by managers on the expected mean returns of the manager benchmark. We show that using a standard model of the SDF from the negative performance puzzle literature, that alpha will be positive when managers add value over the benchmark asset. However, we also show that the sign of alpha is not sufficient for determining investor value, but that alpha needs to exceed some positive threshold to determine whether the existence of managers is value improving for investors. In empirical exercises, we show that this cutoff is non-trivial, and that a number of U.S. domestic equity mutual funds fall on either side of the line.

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1 Introduction

Can an active manager have positive “alpha” but still decrease value for investors? In this paper, we answer yes. Since the pioneering work of Jensen (1968), the evaluation of fund managers by alpha, the intercept in a risk-return regression, has been the industry and academic standard. This research program has found that managers, by and large, have negative alphas. This has led to the negative performance puzzle which asks why investors still allocate money to negative alpha funds. This literature is large. Seminal studies include: Malkiel (1995) who uses one of the first survivorship-bias free databases of funds, Carhart (1997), in partial response to Martin (1996), who finds that a four-factor model can eliminate much of the persistence found in the previous literature, Fama and French (2010) who use a bootstrap procedure and determine luck is a large factor in out-performance, and Barras et al. (2010) who use the False Discovery Rate and show that about 1/5 of funds have negative alpha.[1] However, Kothari and Warner (2001) show in simulation studies that standard measures of out-performance like alpha do not detect out-performance well.

The classic study by Berk and Green (2004) has inspired a literature (Glode (2011), Berk and Van Binsbergen (2015); Berk and van Binsbergen (2017) that seeks to rationalize 0 or negative alphas in models with optimizing investors and managers.[2] The rationality of investing in negative alpha funds does not imply that the existence of these funds is beneficial to investors. This is important. The literature has focused on disproving the claim, roughly that, “if alpha is negative, then the fund is a poor investment.” Our paper shows that the converse is also false: ”If a fund is a poor investment, it must have negative alpha.” That is, we are able to construct a parsimonious equilibrium model where an econometrician would measure positive alpha for all funds. We then show that even positive alpha funds can be value decreasing for investors. This means that the ex ante value function of investors in an equilibrium with funds is less than the ex ante value function when funds do not exist, and the investor only has access to the benchmark.

[1] The results are not uniform. Studies have found positive performance as well, Grinblatt and Titman (1989)

[2] Empirical papers in this vein include: Parwada and Tan (2016) and Leippold and Rueegg (2018).

One may wonder how ostensibly expanding the investor's choice set could lead to lower utility. In a partial equilibrium model, this would indeed be false. However, we impose market clearing a general equilibrium in the benchmark-manager market, reversing the standard intuition.[3] The mechanism works as follows.

The model is a three stage game over any span of time where managers may reset fees or an investor may re-calibrate his or her portfolio. In the first stage of the game, the risk-neutral manager optimally sets his fee to maximize his compensation. Since we assume the fee is a percentage of assets under management (AUM), like an expense ratio, the manager's compensation is simple his fee times the amount of money allocated to him by investors. The manager recognizes that flows respond to fees optimally, so he must trade off setting a high fee, which increases his payoff holding fixed flows, and a low fee, which encourages more flows. He does not, however, take into account the fees of other managers. We seek a symmetric equilibrium where all managers set the same fee. The optimal fee is then a fixed-point of the best response of the manager to market-wide fees.

In the second stage, the investor takes fees, which she knows are identical across managers, and the parameters of the manager's and benchmark's return distribution as given. This stage is a standard portfolio problem for the investor, where we treat the manager as a second risky asset (Garcia and Vanden (2009)), separate from the benchmark, though we allow arbitrary correlation between the two.

Market clearing in the benchmark makes its expected return endogenous. In particular, as the manager sets higher fees, flows that would go to him are instead invested in the benchmark. This increases the current price of the benchmark, and therefore decreases its expected return. This is the key mechanism that allows the investor to be worse-off when having the opportunity to invest in both the manager and the benchmark. Intuitively, if the manager's expected return is sufficiently large, then his existence is beneficial to investors. Otherwise, his return is not large enough to offset the increase in price of the benchmark.

Since most empirical work involves alphas, and not just expected returns, we intro-

[3] Cuoco and Kaniel (2011) also model equilibrium in both markets, but they do not look at out or under-performance.

duce a concept of alpha in this economy. Our definition of alpha is natural. It emerges from a linear stochastic discount factor (SDF) in the benchmark of the manager. This translates to the standard factor models used in the empirical finance literature on performance evaluation. This SDF is misspecified, however. By excluding the manager's own returns from our measure of the SDF, we are missing a component of investor consumption, and hence, marginal utility.[4] Thus, our out-performance does not depend on information differentials within the model as many previous papers require (Admati and Ross (1985), Dybvig and Ross (1985)), or agency problems (Brennan (1993)). Our use of mis-measurement of the SDF to create abnormal returns is in the spirit of Glode (2011). If we take the view that the econometrician is a prospective investor, then we may regard this mis-measurement as a manifestation of the results of Berk and Van Binsbergen (2016) and Barber et al. (2016). These papers show that fund investors typically allocate money according to the CAPM. The missing component of the SDF is naturally positively correlated with the manager returns, and so our measured "betas" will be biased downwards. This leads to positive alpha. It is interesting to note that alphas and fees are positively related in our model. That is, high fee managers are high alpha managers. The intuition is as follows. Managers which are able to set high fees have large enough inflows. Managers with high inflows thus constitute a larger part of the investor's portfolio, making them a larger influence on marginal utility. This exacerbates the mis-measurement of the SDF leading to higher alpha and illustrates the connection between high fees and high alphas.

As mentioned, there is a level of manager mean return above which the manager improves the ex ante utility of the investor. Our definition of alpha is monotone increasing in manager mean return, so we are able to write this cutoff as a function of alpha. That is, we show that there is a level of alpha above which the existence of funds is ex ante value improving for investors. This is particularly interesting since we just established that alpha only exists because of a mis-measured SDF. However, this result says that alpha still can provide information about the performance of managers: If it is large enough, managers are "good" otherwise they are "bad." This cutoff depends on parameters of the

[4]As Roll (1977) argues, this is to be expected. Beber et al. (2018) argue that they typical benchmarks used to evaluate funds do not capture their risk adequately and fairly.

stochastic processes of manager and benchmark returns, as well as investor preferences.

From here, we take our alpha threshold to the data. We demonstrate that this cutoff is not trivial: There are funds above the 50th percentile in alpha that are basically right at the threshold. There are numerous positive alpha funds below it. These results are for US equity mutual funds, and we anticipate that the results will be even stronger for higher-fee vehicles like hedge funds and private equity funds. We perform this exercise in three ways, and all three demonstrate the results are robust. First, we use the theoretical value of alpha to calculate our fund-level alpha. That is, in the model, alpha will be a function of parameters. We use the empirical values of these parameters to estimate alpha without running any regression. In the next two exercises, we follow the literature and regress fund returns on two different benchmarks: the market portfolio a la the Capital Asset Pricing Model (CAPM) (Mossin (1966), Sharpe (1964), Lintner (1965)) and the three-factor model of Fama and French (1993). The figures show the results are similar no matter how we calculate alpha.

Though it is typical to evaluate funds based on how they perform for investors, we may also consider manager and joint manager-investor welfare. Manager value is always positive and so is larger than the (implicit) value of 0 when they do not exist. We define societal welfare the equal-weighted sum of manager and investor value. As just mentioned, manager value is always positive, so this means that we can also formulate an alpha cutoff for increasing/decreasing welfare, but that this cutoff will necessarily be weaker than the previous one. Thus there is a range of alpha where the benefits to managers exceed to losses to investors. As alpha increases, both become "winners." As alpha decreases, the benefits of the managers are no longer enough to offset the investor losses.

Finally, since our framework is so tractable, we include some interesting extensions. First, we consider the case of a large manager. We can think of this case as a situation where there is only one fund manager. Practically speaking, this means there is no difference between his fee and the fees of other managers: he fully internalizes the fact that market wide fees will be the same as his. Second, we consider a case where manager expected returns are a decreasing function of assets under management. This has been suggested as an empirical regularity by Pástor et al. (2015), Tan (2017), and Zhu (2018), among others.

Across both these extensions we find that alpha cutoffs still exist.

The rest of the paper is organized as follows. Section 2 presents the model including the managers' and investors' problems and discusses the equilibrium concept. Section 3 analyses performance measurement and welfare. Data descriptions and empirical findings are in Section 4. Section 5 analyses social welfare. Extensions of the model are discussed in Section 6. Section 7 concludes. Finally, the Appendix contains all omitted proofs.

2 Model

We begin by describing the relationship between a mass one of fund managers and a mass one of rational investors in the economy.[5] The economy lasts three periods, and uncertainty is resolved only at the final period. The time difference between periods one and three can be thought of as any length of time over which investors are likely to re-asses their portfolio holdings.

The model is a sequential game where the investors allocate funds endogenously depending on manager fees and the return processes for the managers and benchmark asset. As opposed to many models in the mutual fund literature (e.g., Berk and Green (2004)), we do not take investor behavior (equivalently, the stochastic discount factor) as exogenous.

We allow for either positive or negative correlation between manager returns and his benchmark portfolio. Instead of postulating a process for the manager's risk adjusted return, "alpha", we derive its sign based on the optimal, rational decisions of the managers and investors. Naturally, alpha will also depend on the first and second moments of returns, and we will be able to derive conditions on these parameters identifying when alpha is positive or negative.

The timing of the model is as follows. In period one, the manager selects the management fee, ϕ, he will charge as a fraction of assets under management (AUM), and a

[5] We use the terms fund manager and active manager interchangeably. Though we test our theory on a sample of U.S. equity mutual funds, our results are applicable to hedge funds, private equity funds, or other similar vehicles.

fund manager and investor are randomly matched.[6] In the second period. the investor optimally chooses her allocation of funds between the manager and the benchmark asset. In the final period, returns are realized.

We will not specify stochastic processes for either the benchmark or the managers' returns. Knowing the first and second moments is sufficient to solve for equilibrium.

2.1 The Investor's Problem

Each investor has the constant absolute risk aversion utility over period three (final period) consumption. Manager payoffs and the benchmark payoff are jointly Normally distributed, and the benchmark is in one unit supply of the numeraire. Thus, maximizing expected utility is equivalent to maximizing the certainty equivalent:

$$\mathbf{E}[C] - (\xi/2)\mathrm{Var}(C) \tag{1}$$

where ξ is the coefficient of absolute risk aversion and governs how much the investor dislikes variance and C is final period consumption. We do not include time-period subscripts, since there is no ambiguity, as the only period with consumption is the final one.

For simplicity, we assume each investor starts with zero wealth. We assume there is a risk-free asset in infinite supply which guarantees the investor a constant gross return R_f. Thus, the budget constraints are:

$$0 = \theta_F \phi + \pi + \theta_f \tag{2}$$

$$C = \theta_F F + \pi R_B + R_f \theta_f \tag{3}$$

Here θ_F is the dollar amount allocated to the fund,[7] ϕ is the fee charged by the fund per dollar invested, π is the dollar amount of the benchmark purchased, θ_f is the dollar

[6] Each manager manages all the flows given to him. Berk and van Binsbergen (2017) show that fund allocation decision within a mutual fund between managers adds substantial value. We do not include this extra step to focus on the core issue.

[7] We could take expectations to be subjective, thereby having funds which investors believe will do well, perhaps based on prior performance be more attractive (Sirri and Tufano (1998)).

amount of the risk-free asset purchased, F is the manager's random return, and R_B is the random return of the benchmark. The fee here should be interpreted as $1 +$ expense ratio. That is, the investor gives the manager θ_F dollars to invest, and for each of those dollars, the manager asks for $\phi\theta_F$ in compensation. So the total expenditure for θ_F dollars given to the manager is $\theta_F * (1+\phi)$. However, for ease of exposition we call $\phi = 1 +$ expense ratio.

We eliminate θ_f from both sides and combine budget constraints:

$$C = \theta_F(F - \phi R_f) + \pi(R_B - R_f) \tag{4}$$

The market clearing condition is:

$$\int_0^1 \pi_i \mathrm{d}i = 1 \tag{5}$$

where i indexes investors. That is, we assume there is one unit outstanding in the numeraire of the benchmark asset. This condition is key, in that models like Berk and Green (2004), Berk and van Binsbergen (2017), and Glode (2011), do not endogenize the price of the alternative risky asset(s)/benchmark. As we will see below, and as Berk and van Binsbergen (2017) note, the fund seeks to maximize $\phi\theta_F$, taking into account the reaction of flows to fees.[8] Adding a market clearing condition can be thought of as adding another mechanism whereby fees affect flows.

Finally, we specify the first and second moments of payoffs. Each manager's mean payoff is $\mathbf{E}[F] = \mu_F$ and his variance is $\mathrm{Var}(F) = \sigma_F^2$. Likewise, $\mathbf{E}[R_B] = \mu_B$ and $\mathrm{Var}(R_B) = \sigma_B^2$. Importantly, due to market clearing in the benchmark, the price of the benchmark is endogenous. Because we write our model in terms of returns, this means both μ_B and σ_B will be endogenous.[9]

We will see that this will manifest itself as a joint condition that must hold in equilibrium between μ_B, σ_B, and other parameters.

The correlation between manager and benchmark returns is γ. We are agnostic about

[8] Our baseline also includes convex costs of managing funds.

[9] The condition will be analogous to the joint condition on expected returns and covariances that must hold in a standard asset pricing model.

how these returns are achieved, but we leave their values unrestricted, allowing us to look at managers who provide hedges for the benchmark portfolio ($\gamma < 0$) and otherwise ($\gamma > 0$).

Managers are ex-ante identical, hence, we seek a symmetric equilibrium where each manager sets the same fee. Therefore, in the second period we do not need to distinguish between the fee a manger charges, ϕ, and the mean/aggregate fee, $\widehat{\phi}$. As we will see in the next subsection, this difference is important from the manager's point of view: He will maximize taking other managers' fees as given.

We also note that since managers are ex-ante identical, there is no gain for diversification among them for investors.

The optimal allocations and joint condition on mean returns and standard deviation of the benchmark in the second period are given in the following proposition.

Proposition 1 *The allocation of each investor to the benchmark and the fund manager she is randomly matched with are:*

$$\pi^* = 1 \tag{6}$$

$$\theta_F^* = \frac{\mu_F - \phi R_f - \xi \sigma_F \sigma_B \gamma}{\xi \sigma_F^2} \tag{7}$$

The expected return of the benchmark must satisfy:

$$\mu_B = R_f + \frac{\gamma \sigma_B}{\sigma_F}\left(\mu_F - \phi R_f\right) + \xi \sigma_B^2 \left(1 - \gamma^2\right) \tag{8}$$

Proof: All proofs are relegated to Appendix A.

For fixed ϕ, allocation to fund manager is, expectedly, decreasing in σ_F ξ, ϕ and R_f and increasing in the mean of the fund return μ_F. Most importantly, the benchmark return is now endogenous because it is decreasing in ϕ. The expected return on the benchmark defines an equation in the price of the asset that must hold in equilibrium.[10]

[10]Note that the fee, ϕ that appears in this Proposition is not agent specific. That is, even though the timing of the model indicates that managers and investors are randomly matched before allocations are decided, we are searching for a symmetric allocation. When the manager decides his fee, he takes all other managers' fees as given, and then symmetry of fees is imposed to find the equilibrium value. When the investors

Notice that the equilibrium condition is similar to a standard asset pricing equation and can be expressed as:

$$\mu_B - R_f = \beta_{B,F}\left(\mu_F - \phi R_f\right) + \xi\sigma_B^2\left(1-\gamma^2\right)$$

Here $\beta_{B,F} = \mathrm{Cov}(R,F)/\mathrm{Var}(F)$. More generally, the above relation is:

$$\mu_B - R_f = -R_f \mathrm{Cov}(m, R_B)$$

where m is a stochastic discount factor (SDF). Looking at the right side of both equations, we see that manager returns and fees will be part of the SDF/marginal utility.

The above fact will be key in performance measurement: Any stochastic discount factor which does not include manager returns will be misspecified. Consequently, the previous expression *does not* hold with equality for *both* the benchmark and the manager's returns. We will construct a simple discount factor (corresponding to the capital asset pricing model (CAPM), for example) which prices the benchmark correctly by design, but will fail to price the manager correctly. This will appear as "alpha."

It is important to note that although we do not directly specify the true stochastic discount factor, the above problem is a standard portfolio choice problem. Consequently, our model still implies no arbitrage (a direct implication of equilibrium). The difference between our model and previous models in the literature (e.g., Glode (2011)), is that our SDF is endogenous. Importantly, it will depend on the fees the managers set through both the price function and through the optimal fund holdings of investors (which in turn affects their equilibrium consumption and, hence, SDF).

2.2 The Manager's Problem

Each active manager seeks to maximize his total payoff which is equal to the fee he charges times the AUM of the fund, $\phi\theta_F$ subject to convex "effort" costs, $-(c/2)\theta_F^2$. Berk and van Binsbergen (2017) note that there is evidence of decreasing returns to scale in asset

are allocating funds, the equality of fees has already been imposed and so all investors face the same fee, regardless of which manager they matched with.

management. Using convex effort costs induces a similar effect and is used in other models, e.g., Glode (2011). The manager anticipates the actions of the investors the next period, and, in particular, recognizes that his AUM will be (endogenously) decreasing in the fee he charges (see Eq.(6)). More formally, the manager maximizes his total payoff subject to the equilibrium conditions of the next period.

We assume managers have no capital of their own, the return technology they possess cannot be traded (but, naturally, they can manage other people's money), and they cannot directly invest in the benchmark independently.

A key difference between our model and Berk and Green (2004), is that manager does not collect all the value he creates. They impose an individual rationality constraint such that investors earn 0 expected excess returns. Of course, by the equilibrium optimality conditions in the second period, the investors are indifferent between allocating to the benchmark or the fund, but this does not imply equivalent expected returns. We only require that the marginal benefit of investing a dollar in the benchmark equal the marginal benefit of investing with a manager including fees.

Finally, as mentioned above, we seek a symmetric equilibrium where all managers charge the same fees. Technically speaking, this means the manager will solve his own problem, taking all other managers' fees as given. We impose the equilibrium concept by ensuring each manager's necessary first-order conditions are satisfied at the symmetric choices for ϕ.

We write the manager's problem as:

$$\max_{\phi;\ \phi \geq 0} \left\{ \phi \cdot \theta_F(\phi, \widehat{\phi}) - \frac{c}{2} \cdot \left(\theta_F(\phi, \widehat{\phi}) \right)^2 \right\} \tag{9}$$

Here c is a parameter governing the disutility the manager from managing a larger fund (e.g., decreasing returns to scale parameter). Notice also that we have written θ_F as a function of the manager's fee, ϕ, which he controls, and $\widehat{\phi}$, the aggregate fee charged by all managers.[11] The allocation depends on $\widehat{\phi}$ through the benchmark price.

[11] Since there is a mass one of managers, $\widehat{\phi}$ is equivalently the mean fee.

Proposition 2 *In a symmetric equilibrium ($\widehat{\phi} = \phi$), each manager's optimal fee is:*

$$\phi^* = \left(\frac{\mu_F - \xi\gamma\sigma_F\sigma_B}{R_f}\right) \cdot \left(\frac{\xi\sigma_F^2(1-\gamma^2) + cR_f}{\xi\sigma_F^2(2-\gamma^2) + cR_f}\right) \tag{10}$$

Naturally, we impose the parameter restriction $\mu_F - \xi\sigma_F\sigma_B\gamma > 0$. Since γ is the correlation between the manager and the benchmark, it is between 0 and 1, and therefore $1-\gamma^2$ and $2-\gamma^2$ are both non-negative.

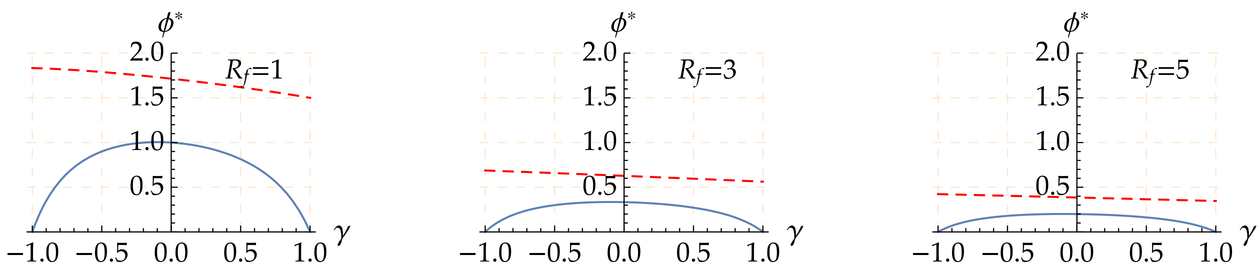


Figure 1: The optimal fee structure w.r.t changing γ, when $c = 0$ (solid, blue line) and $c = 10$ (dashed, red line). Parameters: $\mu_F = 2, \sigma_F = 1, \sigma_B = 0.1$ and $\xi = 2$.

The term $\sigma_F\sigma_B\gamma$ in parentheses is the covariance of the manager with the benchmark. Notice that fee is decreasing in the covariance. As the covariance between the manager and the benchmark increases, the manager provides a worse hedge for investors. As this hedging benefit decreases, the manager is not able to charge as high a fee or else funds would not flow to him. Notice that if $\gamma < 0$, so that the benchmark and manager are negatively correlated, the condition $\mu_F - \xi\gamma\sigma_F\sigma_B > 0$ is always satisfied. There are also ranges of parameters when $\gamma > 0$ that still satisfy this condition. As long as the manager's mean payoff, μ_F, is higher than the penalized covariance, $\xi\text{Cov}(F, R_B)$, the penalization coming from ξ, the condition is satisfied.

When $c = 0$, constant return to scale, the optimal ϕ, which is non-monotone in γ, becomes zero when fund is perfectly correlated with the benchmark return, ie. $\gamma = \pm 1$ (and thus the fund adds no hedging benefit to investors). This is due to the new endogenous pricing effect. In fact, the benchmark asset price adjusts its return so that the excess return due to fund managers goes to zero when fund has no hedging benefit to the investors.

3 Performance Measurement and Welfare

The typical measure of a fund manager's performance is his "Jensen's alpha" or "alpha" for short (Jensen (1968)). In the first subsection, we will derive an expression for manager α. In the second subsection, we will show that α alone is not sufficient to determine whether the existence of fund managers is welfare improving for clients. That is, we will show that there is a range of positive α where the investors' welfare is lower than in a economy without fund managers.

3.1 Manager Alpha

There can be no arbitrage in equilibrium, and, under light regularity conditions, the absence of arbitrage is equivalent to the existence of a strictly positive SDF (for a textbook treatment, see Back (2017)). Let m denote the stochastic discount factor. In principle, we could substitute in the optimal allocations solved for in the previous section and explicitly calculate the SDF as being proportional to marginal utility. However, given that we are in equilibrium, the alpha of any manager assessed this way will be 0.[12]

Since we wish to relate our results to empirical practice, we will take the view of an econometrician or analyst attempting to gauge the manager's performance against a benchmark model. Typically, these benchmarks take the form a linear factor model. A linear factor model for expected returns is generated by an SDF that is affine in the factors: $m = a + b\widetilde{f}$. For simplicity, we take the single factor to be the return of the benchmark over its mean: $\widetilde{f} = B - \mu_B$. Since the gross risk free rate is R_f, we know $\mathbf{E}[m] = a = \frac{1}{R_f}$.

To pin down b, we use the fact that the affine SDF prices the benchmark correctly:

$$\mathbf{E}[mR_B] = \mathbf{E}\left[\left(\frac{1}{R_f} + b(R_B - \mu_B)\right)R_B\right] = 1$$

This implies:

$$b = -\frac{1}{\sigma_B^2}\left[\frac{\mu_B}{R_f} - 1\right] \tag{11}$$

[12]The investor's problem is standard: $\max \mathbf{E}[U(W)]$ subject to a standard budget constraint in two assets: the benchmark and the manager. This leads immediately to $\mathbf{E}[U'(W)(F - R_f\phi)] = 0$.

From this, we can calculate the expected payoff of the manager *under the assumption of a SDF which is affine in the benchmark*. We allow for error, which we call α', in this pricing relationship:

$$\mathbf{E}[mF] = \phi + \alpha' \implies \mathbf{E}\left[\left\{\frac{1}{R_f} - \frac{1}{\sigma_B^2}\left[\frac{\mu_B}{R_f} - 1\right](R_B - \mu_B)\right\}F\right] = \phi + \alpha'$$

Simplifying the above, we get the implied expected return under the SDF:

$$\mu_F = R_f\phi + \beta(\mu_B - R_f) + R_f\alpha' \tag{12}$$

where $\beta = \text{Cov}(F, R_B)/\text{Var}(R) = (\gamma\sigma_F)/\sigma_B$. This is precisely the CAPM implied expected return of the manager. We wish to emphasize that in this simple model, we reach the CAPM. There is nothing special about the benchmark in this case. However, Barber et al. (2016) and Berk and Van Binsbergen (2016) have shown that investors tend to use the CAPM as the model for performance evaluation, i.e., investors seem to consider CAPM alphas as the basis of investment allocation decisions. Suffice to say, we could have regarded R_B as the return on any portfolio which represents a suitable benchmark to the manager and suitable alternative to the manager for investors.

We define manager alpha to be $\alpha \equiv \alpha' R_f$. That is, we define the manager's alpha to be his actual mean excess return minus the theoretical mean excess return we would expect under the SDF which is affine in his benchmark:

$$\alpha = \mu_F - \phi R_f - \beta(\mu_B - R_f)$$

Proposition 3 *The manager's alpha can be written as:*

$$\alpha = \left(\frac{\xi\sigma_F^2(1-\gamma^2)}{\xi\sigma_F^2(2-\gamma^2) + cR_f}\right)(\mu_F - \xi\gamma\sigma_B\sigma_F) \tag{13}$$

Notice that the same restriction which ensures positive fees ($\mu_F > \xi\gamma\sigma_B\sigma_F$) also ensures positive alpha. Thus, alpha is always positive in this model, as measured by the CAPM.

The first obvious thing to note is that for $c = 0$, the constant return to scale, the adjusted fund manager alpha, α, reduces to zero when fund is perfectly correlated, ie. $\gamma = \pm 1$, with the benchmark asset. This is because the endogenous benchmark asset price offset the excess return which is added by the managers and sets fund-fee to zero.

As stated above, under the true SDF, the alpha for all managers is 0. This is similar to Glode (2011). In that paper, the variance of misspecification error in the SDF creates negative alpha. If the SDF were specified exactly in that model, all managers would have 0 alpha. The key difference here is that even our misspecified SDF is endogenous. In particular, the parameter b depends on the price of the benchmark, which is determined in equilibrium.

Notice that, using our earlier expression for the manager's fee:

$$\frac{\alpha}{R_f \phi} = \frac{\xi \sigma_F^2 (1-\gamma^2)}{\xi \sigma_F^2 (1-\gamma^2) + c R_f} < 1$$

Let us consider the extreme case of $c = 0$ (no convex costs for the manager). Then $\alpha = R_f \phi$, the value of an alternative strategy available to the investor: Instead of paying ϕ to the fund, invest it in a risk-free asset. Thus, once we control for the covariance with the benchmark the investor's expected payout from the fund is the same as a risk-free strategy. Recalling that $\alpha = \alpha' R_f$, we also see that $\alpha' = \phi$ in this case: all the misspricing is captured by ϕ.

When $c > 0$, the future value of the fee is larger than alpha. The manager requires extra compensation for managing more money because of his disutility from excess size. This compensation must come from alpha, as the manager cannot change the other parameters governing his expected return or covariance with the benchmark. Indeed, we see the term cR_f in the denominator of the fraction in the expression of alpha, and:

$$\frac{\partial \alpha}{\partial c} = -\frac{\alpha}{\xi \sigma_F^2 (2-\gamma^2) + c R_f} < 0$$

Finally, note that:

$$\frac{\partial \alpha}{\partial \phi} = \left(\frac{\xi \sigma_F (1-\gamma^2)}{\xi \sigma_F (1-\gamma^2) + c R_f} \right) \cdot R_f > 0$$

Recall that this α is as measured by the CAPM/affine SDF model. What this comparative static tells us is that when fund fees are higher the mismeasurement of the true SDF by the CAPM is worse. Fees are endogenous, and two main reasons why we would see higher fees are if a) the mean return of the fund is higher, or b) the variance of benchmark is higher. Both of these statics increase the investor's allocation to the fund manager. So the higher fee is a reflection of higher allocations.

Thus, a higher allocations to the manager imply that marginal utility has a higher covariance with the manager's returns. This is exactly why we see a positive and increasing alpha with respect to fees. Alpha can be regarded as an expression showing the deviation between true mean returns and the means returns as implied by covariance with a candidate SDF. Our candidate SDF has "too small" a covariance, and therefore shows positive alpha.

3.2 Investor Value

We have shown above that if manager fees are non-negative (which they will be in any equilibrium where it is not optimal for the investor to short the manager's technology), then under a misspecified, but empirically standard, risk-adjustment, manager alpha will be non-negative. In this subsection, we will show how to relate manager expected returns to investor value. That is, we will ask the question "Are investors better off with or without fund managers?"

We focus on investor value. In principle one could also look at manager value, and then look at some combination using Pareto weights. We will consider this possibility in the extensions section of the paper. However, because funds are typically evaluated based on whether they provide better investment opportunity tradeoffs for investors[13] we will

[13] Recall that a positive Jensen's alpha implies that an ε-mix of the fund and benchmark mean-variance dominates the benchmark.

only consider investor value here.

Let V_0^* be the ex ante value (period 1 value) of investors in the absence of fund managers. This is the value function of an investor who only has the choice to invest in the benchmark portfolio. Of course, the price of the benchmark portfolio will not be affected by fund fees, since there are no funds to begin with. Let V_1^* denote the ex ante value function of the investor in an economy with fund managers. This is the value function of the investor analyzed in the previous parts of this paper.

The next proposition shows the expressions for the change in value from the no-fund equilibrium to the equilibrium analyzed throughout this paper.

Proposition 4 *The change in value from a no-fund equilibrium to an equilibrium with funds for an investor is given by:*

$$\begin{aligned}\Delta V_1 &\equiv V_1^* - V_0^* \\ &= \frac{\sigma_F \xi \left(\mu_F - \gamma \sigma_F \sigma_B \xi\right)\left(2c\gamma R_f \sigma_B + \sigma_F \left(\mu_F + \gamma\left(3 - 2\gamma^2\right)\sigma_F \sigma_B \xi\right)\right)}{2\left(cR_f + \left(2 - \gamma^2\right)\sigma_F^2 \xi\right)^2}\end{aligned} \tag{14}$$

The sign of the value change hinges on the sign of:

$$2c\gamma R_f \sigma_B + \sigma_F \left(\mu_F + \gamma\left(3 - 2\gamma^2\right)\sigma_F \sigma_B \xi\right)$$

which is the second term in parentheses in the numerator of Eq.(14). Notice that if $\gamma > 0$, then we assured of the positivity of the above expression. When the manager and his benchmark have a positive correlation, we find that a positive alpha is sufficient to ensure that value is improving. This argument hinges on the fact that the investor's allocation to the manager appears in the investor's stochastic discount factor. Consequently, standard factor model regressions to infer fund performance mismeasure the fund's covariance with the SDF. In particular, the manager's returns covary too highly with the SDF in the factor model with the benchmark. This is because (assuming the investor allocates a positive amount to the manager) consumption is increasing in manager returns, and therefore, marginal utility is decreasing in returns. Thus, including the manager return in the SDF should decrease the covariance of manager returns with SDF.

In short, the larger the alpha we find, the more we know the SDF is missing the manager's return. This means, ceteris paribus, that the investor allocated has more funds to that manager in equilibrium.

Next, we assume $\gamma < 0$ for now. The manager and the benchmark have negative correlation. We will express the positivity of the above expression as a restriction on mean manager returns, μ_F. This will provide a simple check for our empirical applications. We could also write everything as a restriction on σ_F. When $\gamma < 0$, a manager is value improving (relative to his non-existence) for investors if:

$$\mu_F > \bar{\mu}_F \equiv -\frac{2c\gamma R_f \sigma_B + \gamma\left(3 - 2\gamma^2\right)\sigma_F^2 \sigma_B \xi}{\sigma_F} > 0 \tag{15}$$

Recall from Eq.(13) that alpha is monotonically increasing in μ_F. Consequently, the above results hold for α in place of μ_F (with proper changes to the right-hand sides). The main takeaway from this being that α is not a sufficient statistic for fund quality, a finding already emphasized in Berk and Green (2004) and Berk and van Binsbergen (2017).

Glode (2011) shows that a misspecified SDF can lead to negatively measured alphas. We have shown that a misspecified SDF can lead to positively measured alphas. Finally, under the correct SDF, alpha should be 0 in our model (and any frictionless equilibrium model). Thus, it should not be surprising that alpha is an insufficient statistic for value analysis. We take this one step further and have shown in a simple model that we can indeed us alpha (manager mean returns) as a value measure. The caveat is that we must go beyond the simple rule $\alpha > 0$ for a "good" fund. In the next section, we take our value cutoff to the data.

It might be surprising that expanding the investor's choice set and lead to lower value. The reason for this is that the choice set is actually altered and not just expanded. Including the fund manager and imposing market clearing means that manager fees impose a pecuniary externality on investors. A higher fee makes the benchmark more appealing, ceteris paribus. Market clearing then raises its price, making the expected return lower.

4 Empirical Results for Negative Correlation Funds

4.1 Data

We use the survivorship bias free mutual fund database from the Center for Research in Security Prices (CRSP). We winnow down our data in the following sequential manner.

First, we focus only on U.S. Domestic Equity funds as identified by CRSP's internal system. Following Glode (2011), we start our sample at 1980 and go through 2017 at a monthly frequency.

Second, since most funds do not have existing data for the full sample period, we set a minimum number of consecutive months of reported data. For our main result, we set this lower bar at 36, though we experimented with more stringent 100 months as well.

Finally, we removed all funds with missing values for net asset value (NAV). This leaves us with a sample of 12,697 funds.

We download benchmark excess return and risk-free rate data from Kenneth French's website, since we will follow the model and assume the benchmark is the benchmark portfolio a la the CAPM. The next section describes our empirical test.

4.2 Value Comparisons

We start by translating our bound developed in the previous section into a bound on alpha. Use Eq.(13) to write μ_F as a function of α. Then, the bound on α in order for investor value to be higher in the presence of funds is now:

$$\overline{\alpha} = -2\gamma(1-\gamma^2)\sigma_F\sigma_B\xi \tag{16}$$

When $\gamma > 0$, we see that $\alpha > \overline{\alpha}$, since $\overline{\alpha}$ is negative and we have shown that $\alpha > 0$ in our model. This bound has bite, then, when $\gamma < 0$. This is surprising, since $\gamma < 0$ corresponds to the case where the manager provides a hedge to the benchmark. The reason is related to our discussion of fees and alpha above. The manager sets his fee so, after taking into account his own convex costs, the investor is indifferent between allocating a dollar to versus not doing so. The higher fees manifest themselves as higher alpha according to the

CAPM.

Alpha can be positive either because the manager is earning high mean returns, or if the CAPM assigns low expected returns to the manager. Only the former explanation should be value increasing. The bound $\overline{\alpha}$ tells precisely how high alpha must be in order for the former effect to dominate the latter.

We set $\xi = 2$ and $c = 5$.[14] The other parameters in the bounds (including in the definition of α) are derived from the fund manager's data or the benchmark return. For R_f, which appears as a parameter in our model, we use the mean risk-free rate over 1980-2017.

Lastly we restrict our sample to funds with $\gamma < 0$ and $\mu_F > 0$. This reduces our sample to 2,533. This is not surprising, since most equity mutual funds will have positive correlations with the benchmark. Table (1) shows summary statistics for these funds. Note that the mean fund size is far larger than the median. This is not unexpected given previous studies. Table (2) summarizes the fixed parameter values mentioned above.

Table 1: Summary Statistics

	Mean	Median	S.D.	25th Pct.	75th Pct.
Assets ($M)	278.3322	15.1	1300.7641	1.7	88.3
NAV	21.4499	15.39	19.7856	12	23.3
Expense Ratio (%)	1.0644	1.0243	0.6316	0.633	1.43
Return (%, Monthly)	0.6404	0.8999	4.0267	-1.51	2.99

This table shows the summary statistics for 2,533 funds used to construct the figures. Expense ratios are reported quarterly and it is assumed they are constant over each month in the quarter. Return is calculated as NAV_t/NAV_{t-1}.

Table 2: Parameters Used in Construction of Value Bounds

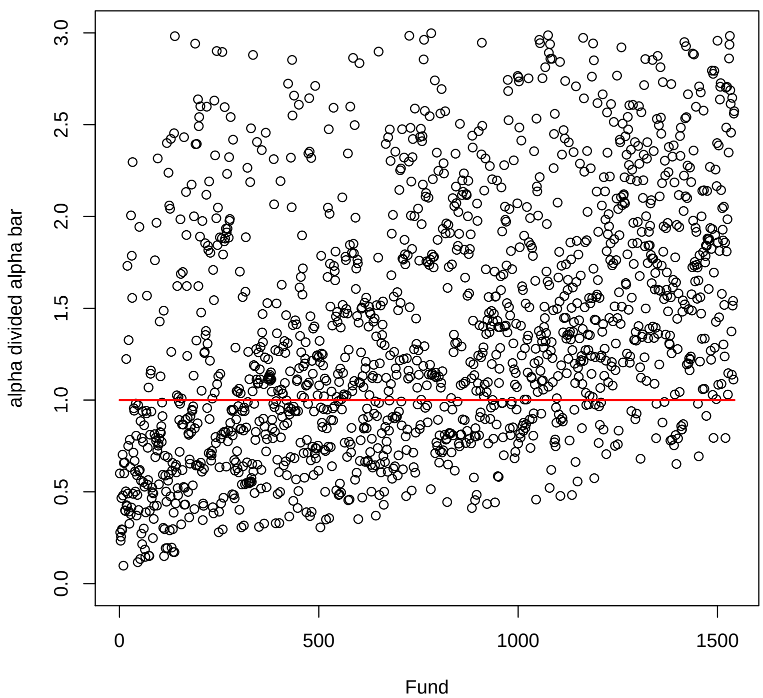
Risk-Free Rate (%, Monthly)	0.2746
ξ	2
σ_B (%, Monthly)	5.33
c	5

Figure (2) displays the difference $\alpha_f/\overline{\alpha}_f$ for each fund. We include the subscript f to emphasize that each variable is fund specific. Funds with points above one are value

[14] The results are not sensitive to the choice of c.

improving, and funds with points below are not. The funds are sorted by their alpha (as computed by Eq.(13)), so that dots on the left correspond to lower alpha funds, and dots to the right, higher alpha ones. While the mass of points does move up and to the right, we still see a number of "high alpha" fund with $\overline{\alpha}$ below 1, indicating that these funds do not improve value for investors.[15]

Figure 2: Value Bounds for Funds



This figure displays the ratio $\alpha_f/\overline{\alpha}_f$ for each fund f based on the theoretical values for α_f and $\overline{\alpha}_f$.. Each circle corresponds to a fund. The funds are ordered along the x-axis in order of increasing alpha. That is, the circles furthest to the right have the largest α_f. The red horizontal line indicates the theoretical cutoff, above which funds are value increasing for investors.

4.3 Empirical Welfare Bounds

In the previous subsection, we computed the value bound using the theoretical values of alpha from the model. That is, we computed alpha as a function of fund and investor

[15]It is known that the CRSP database biases results upward for funds, (Elton et al. (2001)), so this might be a upper bar on performance.

parameters. In this subsection, we follow the standard fund performance literature and estimate alpha empirically using the CAPM . Conceptually, the two exercises are very similar. The main difference comes from how fees are handled. In the theoretical implementation, fees disappeared from the computation of alpha because fees are themselves functions of firm and investor parameters. This model is not meant to quantitatively capture the exact fee-setting mechanisms of managers, so computations of alphas using empirical measures of fund fees will be different than our theoretical estimates.

Recall from Eq.(4) (the budget constraint) that ϕ is the cost per dollar allocated to the fund. In our mutual fund data, we calculate this as 1 plus the expense ratio, which is in percentage of net asset value. This is in line with the model interpretation, for if θ_F is the dollar amount invested, then the investor hands over an extra $\theta_F *$ expense ratio to the manager, so $\phi = (1 + \text{expense ratio})$. Fund expense ratios are reported quarterly in CRSP.

Even though the model is not truly dynamic, we estimate the following model:

$$R_{i,t} - \phi_i R_f = \alpha_i + \beta_i (R_{M,t} - R_f) + \varepsilon_{i,t} \tag{17}$$

where $\varepsilon_{i,t}$ is mean zero error and i indexes managers. We write ϕ and R_f as constants so that when the unconditional expectation is passed through Eq.(17) we have

$$\mu_{R_i} - \phi_i R_f = \alpha_i + \beta_i (\mu_M - R_f)$$

which corresponds to our definition of alpha in the model. One issue is that both R_f and ϕ are time-varying, though modestly, in the data. We can replace the R_f on the right-hand side by the $\mathbf{E}[R_f]$ when we pass the expectation operator through. However, following this reasoning leads to the term $\mathbf{E}[\phi R_f]$. To be consistent, we replace this term with $\mathbf{E}[\phi]\mathbf{E}[R_f]$. Eq. (17) is estimated for each fund, and we collect the alphas.

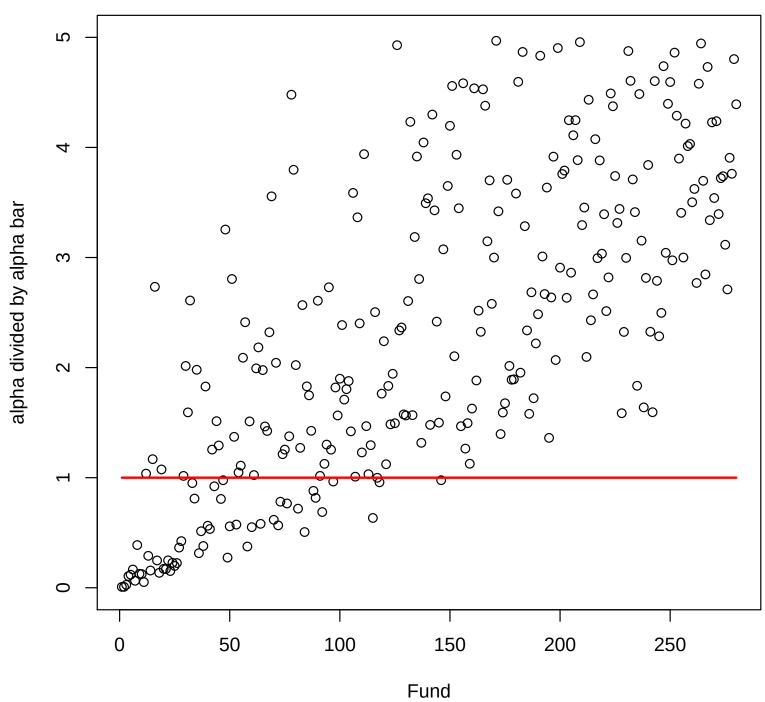
The sample of funds is the same as in the previous subsection, that is, we have 2,533 funds reporting for at least 36 consecutive months from 1980 through 2017. Also, these funds all have positive average returns and negative correlations with the benchmark return.

Running regression (17), we collect the estimated α_i's and compute $\overline{\alpha}_i$ for each fund.

We winsorize $\overline{\alpha}_i$ at 0 and 5 to focus on funds near the 1 cutoff. This further reduces our sample to 333 funds.[16]

Figure (3) plots $\overline{\alpha}_i$ for funds against the value bound. Unsurprisingly, lower α funds (the funds are sorted on alpha again) tend to be below the cutoff, implying that they there are not ex ante value improving for investors. Similarly, high alpha funds tend to be above the cutoff signaling they are value improving for investors. Note, however, that even funds above the median alpha (index number above ≈ 150 are on or near the cutoff line.

Figure 3: Welfare Cutoffs for Funds Using CAPM Alphas



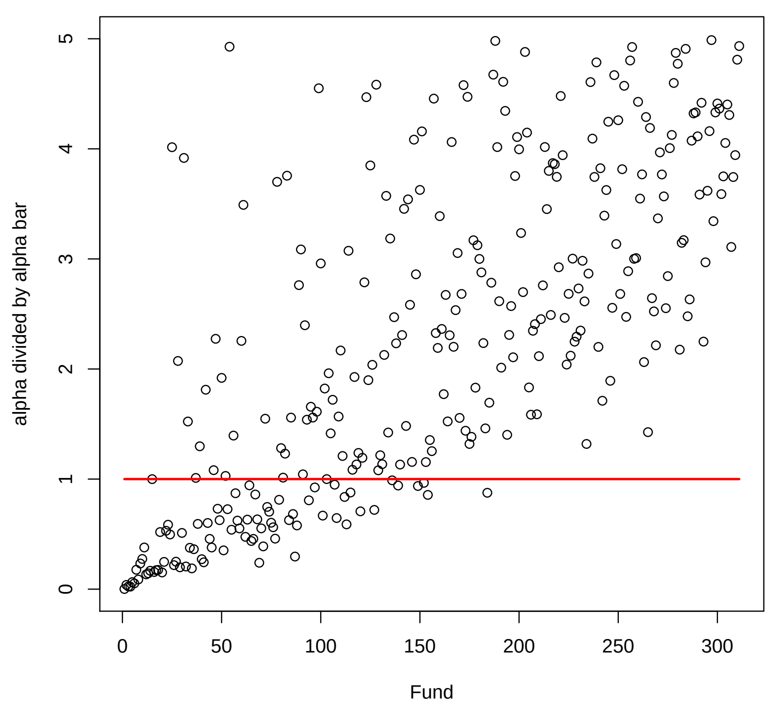
This figure displays the ratio $\alpha_f / \overline{\alpha}_f$ for each fund f. Here α_f is the estimated CAPM alpha for each fund.The funds are ordered along the x-axis in order of increasing alpha. That is, the circles furthest to the right have the largest α_f. The red horizontal line indicates the theoretical cutoff, above which funds are value increasing for investors.

While the model is written with one benchmark asset, this simply for simplicity. There is nothing that prevents from assuming that the benchmark asset is a portfolio (like the market portfolio) or a combination of portfolios. To to that end, we use the popular

[16]The majority of funds were either far above or below.

Fama-French three factor model (Fama and French 1993, 1996) as our benchmark for a robustness test. Aside from the excess return on the market over the risk-free rate, there is also a portfolio of small market capitalization stocks over large ones, and a portfolio of high book-to-market value stocks over small ones. As in the above exercise, we consider the alpha for each fund when we use the FF3 model. This is simply the intercept from a regression of fund excess returns on the factors and a constant. The results from Figure (5) show that our results are robust to benchmark changes. Qualitatively, almost nothing changes. There are, in fact, more funds near the cutoff line. This is not unsurprising: We are controlling for more sources of risk, which have shown to lower alphas for funds, in general.

Figure 4: Welfare Cutoffs for Funds Using FFs Alphas



This figure displays the ratio $\alpha_f / \overline{\alpha}_f$ for each fund f. Here α_f is the estimated Fama-French 3 factor alpha for each fund. The funds are ordered along the x-axis in order of increasing alpha. That is, the circles furthest to the right have the largest α_f. The red horizontal line indicates the theoretical cutoff, above which funds are value increasing for investors.

5 Social Welfare

In a previous section, we showed that positive alpha is not indicative of a value improvement for investors, but that a sufficiently high alpha is. In this subsection we will address the same idea for fund managers and for the whole economy (i.e., managers plus investors).

Let V_M^* denote the ex ante value of a fund manager operating a fund.[17] Recall that V_0^* is investor value in an economy without funds and V_1^* is the investor value in an economy with funds.

The next proposition shows the expressions for the change in welfare from the no-fund equilibrium to the equilibrium analyzed throughout this paper.

Proposition 5 *For the fund manager, the change in value is given by:*

$$V_M^* = \frac{\left(cR_f - 2(\gamma^2-1)\sigma_F^2\xi\right)(\mu_F - \xi\gamma\sigma_R\sigma_F)^2}{2R_f\left(cR_f - (\gamma^2-2)\,\sigma_F^2\xi\right)^2} > 0 \tag{18}$$

For the economy as a whole, that is, $V_1^* - V_0^* + V_M^*$*, the change in value is given by*

$$\begin{aligned}\Delta V_S &\equiv V_M^* + \Delta V_1 \\ &= \Big[\left(\mu_F - \gamma\sigma_F\sigma_B\xi\right)\Big(cR_f\left(\mu_F + \gamma(2R_f-1)\sigma_F\sigma_B\xi\right) + \sigma_F^2\xi\Big(\mu_F\left(2-2\gamma^2+R_f\right) \\ &\quad + \gamma\left(3R_f - 2\gamma^2(R_f+1)-2\right)\sigma_F\sigma_B\xi\Big)\Big)\Big]\Big/\Big[2R_f\left(cR_f - \left(\gamma^2-2\right)\sigma_F^2\xi\right)^2\Big]\end{aligned} \tag{19}$$

First, consider the middle term, Eq.(18). As indicated, this is always positive, since $-1 < \gamma < 1$. Therefore, managers always benefit by opening a fund.

The sign of ΔV_S is the same as as the sign of:

$$\begin{aligned}&cR_f\left(\mu_F + \gamma(2R_f-1)\sigma_F\sigma_B\xi\right) + \sigma_F^2\xi\Big(\mu_F\left(2-2\gamma^2+R_f\right) \\ &+ \gamma\left(3R_f - 2\gamma^2(R_f+1)-2\right)\sigma_F\sigma_B\xi\Big)\end{aligned}$$

[17] As stated in a previous footnote, we assume managers have 0 utility when they do not "exist."

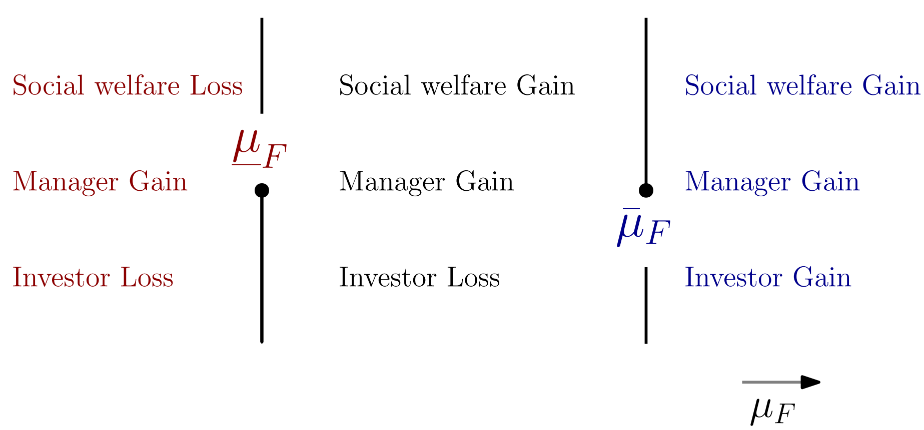
Rearranging this as a restriction on μ_F yields:

$$\mu_F > \underline{\mu}_F \equiv \frac{\gamma\sigma_F\sigma_B\xi\left(c\left(1-2R_f\right)R_f + \left(2+2\gamma^2\left(R_f-1\right)-3R_f\right)\sigma_F^2\xi\right)}{cR_f + \left(2-2\gamma^2+R_f\right)\sigma_F^2\xi} \tag{20}$$

Eq.(20) is a weaker bound that Eq.(15). This is intuitive, since ΔV_S includes the manager's welfare, which we know is always positive.

Thus, there are intermediate levels of alpha (or μ_F) where the gain to managers is large enough to offset the loss to investor value. Once alpha gets even higher, "everybody wins." For alpha below the cutoff implied Eq.(20), the gain to managers is not enough to offset the losses to investors. The cutoff implied by Eq.(20) is smaller than the cutoff implied by Eq.(15), though both are greater than zero. We believe the cutoff proposed in the previous sections is more in-line with academic and practitioner thinking. Typically, when evaluating fund quality, we are concerned with how the fund performs from the investor's perspective and do not care about the benefits the manager generates for himself by running his fund.

Figure 5: Investor, manager and welfare cutoffs



This figure displays the cutoffs for the managers, the investors and the social welfare. When $\mu_F < \underline{\mu}_F$ only managers gain. When $\underline{\mu}_F \leq \mu_F \leq \bar{\mu}_F$ still managers gain and investors loss, however, manager's gain is large enough so that the net social welfare is positive. Finally, when $\mu_F > \bar{\mu}_F$, *every body wins*, i.e., both managers and investors gain leading to positive gain in welfare as well.

6 Extensions and Robustness

In this section we extend our model in other directions and analyze the robustness of our main findings. In Section 6.1 we show all the main results hold when there is only a monopolistic manager. In Section 6.2 we show that our main conclusions do not depend on deliberate functional forms and can go beyond quadratic convex cost in AUM.

6.1 A Large Manager

In this subsection, we consider the case where the fund manager is large. This means that he understands that the fee he sets will be the prevailing fee in the benchmark. More simply put, recall that $\mu_F\left(\widehat{\phi}\right)$ is a function of the aggregate fee, $\widehat{\phi}$. When the manager is large (e.g., a monopoly), we consider $\mu_F(\phi)$ and allow the manager to optimize his fee taking into account both its direct effect on θ_F as well as its effect on μ_B.

The portfolio problem of the investors is unchanged, so we do not restate those results.

Proposition 6 *A large manager's optimal fee is:*

$$\phi_L^* = \left(\frac{\mu_F - \gamma\sigma_F\sigma_B\xi}{R_f}\right)\left(\frac{\sigma_F^2\xi + cR_f}{2\sigma_F^2\xi + cR_f}\right) \tag{21}$$

Recall from our main specification that:

$$\phi^* = \left(\frac{\mu_F - \gamma\sigma_F\sigma_B\xi}{R_f}\right)\left(\frac{\sigma_F^2\xi\left(1-\gamma^2\right) + cR_f}{\left(2-\gamma^2\right)\sigma_F^2\xi + cR_f}\right)$$

and one can show that:

$$\phi_L^* > \phi^*$$

There are three effects going on. First, a higher fee induces less inflows, θ_F. Second, a higher fee increases $\phi\theta_F$ for fixed θ_F. Third, a higher fee increases the price of the benchmark and, therefore, decreases its expected return, μ_B. For $\phi \in (\phi^*, \phi_L^*)$, the net effect of these three forces increases the manager's value function.

We state without proof (it is identical to the analogous proof above) the value bounds for when $\gamma < 0$ in the case of a large fund manager:

$$\Delta V_1^* > 0 \iff \alpha > \frac{2\gamma\left(\gamma^2-1\right)\sigma_F\sigma_B\xi\left(cR_f+2\sigma_F^2\xi\right)}{cR_f-\left(\gamma^2-2\right)\sigma_F^2\xi} \tag{22}$$

6.2 Decreasing Returns to Scale

Following Berk and van Binsbergen (2017), we change the specification of manager returns to:

$$F = a - \frac{b}{2}\theta_F + \varepsilon_F \tag{23}$$

where $\varepsilon_F \sim N\left(0, \sigma_F^2\right)$ and $a, b > 0$. This implies:

$$\mu_F = a - \frac{b}{2}\theta_F \tag{24}$$

We can look at this formulation of returns as an alternative to the convex costs imposed earlier. Our earlier method decreases the manager's payoff once θ_F gets too large, so the manager sets his fee taking this into account. In the current method, the investors understand that the more they invest with the manager, the lower their expected return. Both methods should decrease θ_F versus a setting with no costs or decreasing returns to scale.

Empirically (Pastor, Stambaugh, and Taylor, 2015) there is evidence of decreasing returns to scale in fund management. It is important to note that our main results do not hinge on whether we include convex costs or decreasing returns to scale, as we now show.

Proposition 7 *When there are decreasing returns to scale in fund management, the investor's optimal portfolio allocations are given by:*

$$\theta_F^* = \frac{\left(a-\phi R_f\right)\sigma_B - \left(\mu_B - R_f\right)\sigma_F\gamma}{\sigma_B\left[b+\xi\sigma_F^2\left(1-\gamma^2\right)\right]} \tag{25}$$

$$\pi^* = 1 \tag{26}$$

The market clearing condition requires the following equality to hold:

$$\mu_B = R_f + \left(\frac{\left(a - \phi R_f\right) \xi \sigma_B \sigma_F \gamma + \xi \sigma_B^2 \left[b + \xi \sigma_F^2 \left(1 - \gamma^2\right)\right]}{b + 2\xi \sigma_F^2 \left(1 - \gamma^2\right)} \right) \tag{27}$$

Since there are decreasing returns to scale in fund returns, we do not additionally impose the convex costs on the manager. Thus, the manager solves:

$$\max_{\phi;\ \phi \geq 0} \{\phi \cdot \theta_F\}$$

As before, we seek a symmetric equilibrium.

Proposition 8 *The manager's optimal fee in a symmetric equilibrium in fees is:*

$$\phi^* = \frac{a\left(b + \left(2 - 3\gamma^2\right)\sigma_F^2 \xi\right) + \gamma \sigma_F \sigma_B \xi \left(-b + \left(-1 + \gamma^2\right)\sigma_F^2 \xi\right)}{2bR_f + \left(4 - 5\gamma^2\right) R_f \sigma_F^2 \xi} \tag{28}$$

Analyzing the positivity of this solution is more complicated, but algebraic manipulations show that there are many parameter combinations where this expression is positive.

Evaluating the value bound is more complicated, as now μ_F is a function of θ_F and the parameters a, b. One can show that there do exist values of b, for fixed $a > 0$, where $V_0 - V_1 > 0$ and $V_0 - V_1 < 0$. Thus, the value bound exists.

7 Conclusion

We have presented a parsimonious equilibrium model of optimal portfolio choice by investors and fee setting by active managers. In our model, true alpha is always zero, since it must be in equilibrium. However, using standard performance evaluation methods from the empirical literature, an econometrician would assign positive alpha to all funds in our economy. This result emerges because the econometrician is using a misspecified SDF.

Since our model is an equilibrium model, we are able to make normative statements. We show that even though the sign of alpha is not sufficient to judge fund quality, there is a positive threshold of alpha which can determine fund quality in our model. We show that this threshold is non-trivial empirically. That is, among the set of US equity mutual funds, with certain characteristics, there are positive alpha funds that are not value increasing for investors. We view these results as a lower-bound: We hypothesize the empirical results will be more stark with more exotic and higher-fee investment vehicles.

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A Appendix

A.1 Proofs

Proof of Proposition 1: The investor's problem is:

$$\max_{\theta_F,\pi} \mathbf{E}[C] - \frac{1}{2}\text{Var}(C) \text{ subject to } W_0 = \theta_F\phi + \pi + \theta_f,\ C = \theta_F F + \pi R_B + \theta_f R_f,\ W_0 = 0$$

Replace W_0 with 0 in the first constraint and substitute out θ_f, the allocation to the risk-free asset between the two remaining constraints. This yields:

$$C = \theta_F\left(F - \phi R_f\right) + \pi\left(R_B - R_f\right)$$

Now, we can compute the following moments of consumption:

$$\begin{aligned}\mathbf{E}[C] &= \theta_F(\mu_F - \phi R_f) + \pi(\mu_B - R_f)\\ \text{Var}(C) &= \theta_F^2\sigma_F^2 + \pi^2\sigma_B^2 + 2\theta_F\pi\sigma_B\sigma_F\gamma\end{aligned}$$

These moments imply that the investor's problem is:

$$\max_{\theta_F,\pi}\left\{\theta_F(\mu_F - \phi R_f) + \pi(\mu_B - R_f) - \frac{\xi}{2}\theta_F^2\sigma_F^2 - \frac{\xi}{2}\pi^2\sigma_B^2 - \xi\theta_F\pi\gamma\sigma_B\sigma_F\right\}$$

The first-order conditions (which are necessary and sufficient by concavity) are, for θ_F and π, respective:

$$\begin{aligned}\theta_F: &\quad (\mu_F - \phi R_f) - \xi\sigma_F^2\theta_F - \xi\pi\gamma\sigma_B\sigma_F = 0\\ \pi: &\quad (\mu_B - R_f) - \xi\sigma_B^2\pi - \xi\theta_F\gamma\sigma_B\sigma_F = 0\end{aligned}$$

Multiply the second equation by $-\frac{\gamma\sigma_F}{\sigma_B}$ and add it to the first equation to get:

$$(\mu_F - \phi R_f) - \frac{(\mu_B - R_f)\gamma\sigma_F}{\sigma_B} - \xi\sigma_F^2\theta_F + \xi\theta_F\gamma^2\sigma_F^2 = 0$$

Rearranging, we find:

$$\theta_F^* = \frac{(\mu_F - \phi R_f)\sigma_B - (\mu_B - R_f)\gamma\sigma_F}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}$$

We assume the parameters are such that the numerator is nonnegative. Otherwise, the manager's mean return is sufficiently low so as to induce investors to short him. This problem is relatively uninteresting in this model, so we ignore it.

Use this solution in the FOC for π and rearrange to find:

$$\pi^* = \frac{(\mu_B - R_f)\sigma_F - (\mu_F - \phi R_f)\sigma_B\gamma}{\xi\sigma_B^2\sigma_F(1-\gamma^2)}$$

We also require this the numerator of this expression to be positive, since otherwise the required market clearing condition cannot hold, and equilibrium does not exist.

Now we use the market clearing condition that there is one unit outstanding of the benchmark. We also use the fact that we are seeking a symmetric equilibrium. That is, we have written the fee charged by the manager the investor matched with as ϕ, as opposed to ϕ_i. This is important, since at this stage in the game, the investor is confronted with an identical menu of fees. As we will see, this is not the case when we solve the manager's problem. Consequently, nothing on the right side of the above expression is investor i specific. This implies when we integrate over i, that:

$$(\mu_B - R_f)\sigma_F - (\mu_F - \phi R_f)\sigma_B\gamma = \xi\sigma_B^2\sigma_F\left(1-\gamma^2\right)$$

Solving for expected returns:

$$\mu_B = R_f + (\mu_F - \phi R_f)\frac{\sigma_B\gamma}{\sigma_F} + \xi\sigma_B^2\left(1-\gamma^2\right)$$

Note that prices are endogenous, even if they are not explicitly modeled here. That means, the above expression is an implicit equation in the price, which would appear, for example, in the denominator of μ_B. The above condition is a joint restriction on μ_B and σ_B which must hold in equilibrium.

Substituting this back in to the solution for π, we see:

$$\pi^* = 1$$

Finally, substituting the equilibrium expected back in to the solution for θ_F^*:

$$\theta_F^* = \frac{\mu_F - \phi R_f - \xi\sigma_F\sigma_B\gamma}{\xi\sigma_F^2}$$

□

Proof of Proposition 2: Recall the optimal θ_F as a function of μ_B from the previous proof:

$$\theta_F = \frac{\sigma_B(\mu_F - \phi) - \gamma\sigma_F(\mu_B(\widehat{\phi}) - R_f)}{\xi\sigma_B\sigma_F^2(1-\gamma^2)}$$

where we have written μ_B as a function of the mean $\widehat{\phi}$.[18]. It is important to note that the manager does not have any market power, and, although in equilibrium we seek a symmetric solution where all ϕ's are the same, the manager takes the "market fee" as given. In effect, what we do below is find a fixed point where the manager's best response to a market fee of ϕ is exactly ϕ as well. Therefore, the manager solves:

$$\max_{\phi\geq 0}\left\{\phi\theta_F(\phi,\widehat{\phi}) - \frac{c}{2}\theta_F^2(\phi,\widehat{\phi})\right\} \tag{29}$$

The first-order condition for this problem is:[19]

$$\begin{aligned}&\frac{(\mu_F - 2R_f\phi)\sigma_B - (\mu_B(\widehat{\phi}) - R_f)\gamma\sigma_F}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\\ &\quad + c\left[\frac{(\mu_F - R_f\phi)\sigma_B - (\mu_B(\widehat{\phi}) - R_f)\gamma\sigma_F}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\right]\left(\frac{R_f\sigma_B}{\xi\sigma_B\sigma_F^2(1-\gamma^2)}\right) = 0\end{aligned}$$

Replace $\widehat{\phi}$ with ϕ, enforcing the symmetric equilibrium, and use our previous expression

[18]Since there is mass one of managers, this is also the aggregate fee.
[19]The problem is concave in ϕ, so the FOC is also sufficient.

for μ_B. Multiply through by $\xi\sigma_F^2\sigma_B(1-\gamma^2)$:

$$(\mu_F - 2R_f\phi)\sigma_B - \left((\mu_F - \phi R_f)\frac{\sigma_B\gamma}{\sigma_F} + \xi\sigma_B^2\left(1-\gamma^2\right) - R_f\right)\gamma\sigma_F$$
$$+ cR_f\sigma_B\left[\frac{(\mu_F - R_f\phi)\sigma_B - \left((\mu_F - \phi R_f)\frac{\sigma_B\gamma}{\sigma_F} + \xi\sigma_B^2\left(1-\gamma^2\right)\right)\gamma\sigma_F}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\right] = 0$$

Rearranging and cancelling terms yields:

$$\phi^* = \left(\frac{\mu_F - \xi\gamma\sigma_F\sigma_B}{R_f}\right)\left(\frac{\xi\sigma_F^2(1-\gamma^2) + cR_f}{\xi\sigma_F^2(2-\gamma^2) + cR_f}\right)$$

We require this expression to be nonnegative, as an equilibrium with negative fees is uninteresting. □

Proof of Proposition 3: Our definition of alpha is:

$$\alpha = \mu_F - R_f\phi - \frac{\sigma_F\gamma}{\sigma_B}(\mu_B - R_f)$$

Simply, plug in expressions for ϕ and μ_B and rearrange to find:

$$\alpha = \left(\frac{\xi\sigma_F^2(1-\gamma^2)}{\xi\sigma_F^2(2-\gamma^2) + cR_f}\right)[\mu_F - \xi\gamma\sigma_B\sigma_F]$$

□

Proof of Proposition 4: First, we find V_0. The investor's problem here is standard for mean-variance preferences/CARA utility, and we follow the exact same steps as the Proof of Proposition 1. The relevant moments are:

$$\mathbf{E}[C] = \pi(\mu_B - R_f)$$
$$\mathrm{Var}(C) = \pi^2\sigma_B^2$$

Therefore, the problem is:

$$\max_{\pi} \pi(\mu_B - R_f) - \frac{\xi}{2}\pi^2\sigma_B^2$$

The solution to this problem is:

$$\pi = \frac{\mu_B - R_f}{\xi \sigma_B^2}$$

Integrate the left-hand side over all investors (noting that the right-hand side is the same for all of them) to find the market clearing mean return:

$$\mu_B = R_f + \xi \sigma_B^2$$

As expected, this means:

$$\pi^* = 1$$

Finally, inserting the equilibrium values for μ_B and π in the value function yields:

$$V_0^* = \frac{\xi \sigma_B^2}{2}$$

We follow the same general steps as above to derive V_1^*. That is, we evaluate

$$\begin{aligned}\mathbf{E}[C] &= \theta_F(\mu_F - \phi R_f) + \pi(\mu_B - R_f)\\ \mathrm{Var}(C) &= \theta_F^2 \sigma_F^2 + \pi^2 \sigma_B^2 + 2\theta_F \pi \sigma_B \sigma_F \gamma\end{aligned}$$

at the equilibrium values of θ_F, π, μ_B, and ϕ (see previous propositions). Then, $V_1^* = \mathbf{E}[C] - (\xi/2)\mathrm{Var}(C)$.

The value change, $V_1^* - V_0^*$ is, after a great deal of tedious algebra:

$$\begin{aligned}\Delta V_1 &\equiv V_1^* - V_0^*\\ &= \frac{\sigma_F \xi \left(\mu_F - \gamma \sigma_F \sigma_B \xi\right) \left(2c\gamma R_f \sigma_B + \sigma_F \left(\mu_F + \gamma \left(3 - 2\gamma^2\right) \sigma_F \sigma_B \xi\right)\right)}{2\left(cR_f - (\gamma^2 - 2)\, \sigma_F^2 \xi\right)^2}\end{aligned}$$

The sign of this expression depends on the sign of the second term in parentheses. □

Proof of Proposition 5: The manager's ex ante wealth in the no-fund equilibrium is set to 0. Define V_M^* to be the manager's ex ante welfare in the equilibrium described in the paper. It's value is:

$$V_M^* = \phi\theta_F - \frac{c}{2}\theta_F^2 = \frac{\left(cR_f - 2(\gamma^2-1)\sigma_F^2\xi\right)(\mu_F - \xi\gamma\sigma_B\sigma_F)^2}{2R_f\left(cR_f - (\gamma^2-2)\,\sigma_F^2\xi\right)^2} > 0$$

where we have evaluated the expression at the equilibrium levels of ϕ and θ_F. Manager welfare is always higher (greater than zero) in the equilibrium in which funds exist.

Lastly, define societal welfare as $V_M^* + \Delta V_1$. Some tedious algebra leads to:

$$\begin{aligned}\Delta V_S &\equiv V_M^* + \Delta V_1 \\ &= \Big[\Big(\mu_F - \gamma\sigma_F\sigma_B\xi\Big)\Big(cR_f\Big(\mu_F + \gamma(2R_f-1)\sigma_F\sigma_B\xi\Big) + \sigma_F^2\xi\Big(\mu_F\Big(2-2\gamma^2+R_f\Big) \\ &+ \gamma\left(3R_f - 2\gamma^2(R_f+1)-2\right)\sigma_F\sigma_B\xi\Big)\Big)\Big]\Big/\Big[2R_f\left(cR_f - \left(\gamma^2-2\right)\sigma_F^2\xi\right)^2\Big]\end{aligned}$$

The sign of this expression is determined by the sign of the second term in parentheses in the numerator. □

Proof of Proposition 6: The manager solves the following problem:

$$\max_{\phi;\ \phi\geq 0}\left\{\phi\theta_F(\phi) - \frac{c}{2}\theta_F^2(\phi)\right\}$$

Note that now θ_F is only a function of ϕ and not both ϕ and $\widehat{\phi}$. This is because he fully internalizes that he is the only manager and therefore his fee is the "market fee." Plugging in for θ_F we have:

$$\begin{aligned}\max_{\phi;\ \phi\geq 0}\Bigg\{\phi\cdot\left(\frac{(\mu_F-\phi R_f)\sigma_B - (\mu_B(\phi)-R_f)\gamma\sigma_F}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\right) \\ - \frac{c}{2}\cdot\left(\frac{(\mu_F-\phi R_f)\sigma_B - (\mu_B(\phi)-R_f)\gamma\sigma_F}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\right)^2\Bigg\}\end{aligned}$$

Next, recall that, in equilibrium:

$$\mu_B(\phi) - R_f = (\mu_F - \phi R_f)\frac{\sigma_B\gamma}{\sigma_F} + \xi\sigma_B^2\left(1-\gamma^2\right)$$

Using this in the above maximization problem gives:

$$\max_{\phi;\ \phi\geq 0}\left\{\phi\cdot\left(\frac{(\mu_F-\phi R_f)\sigma_B-(\mu_F-\phi R_f)\gamma^2\sigma_B-\xi\sigma_B^2\sigma_F\gamma\left(1-\gamma^2\right)}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\right)\right.$$
$$\left.-\frac{c}{2}\cdot\left(\frac{(\mu_F-\phi R_f)\sigma_B-(\mu_F-\phi R_f)\gamma^2\sigma_B-\xi\sigma_B^2\sigma_F\gamma\left(1-\gamma^2\right)}{\xi\sigma_F^2\sigma_B(1-\gamma^2)}\right)^2\right\}$$

The solution to this problem once again requires $\mu_F-\xi\sigma_B\sigma_F\gamma>0$ in order for $\phi>0$. Following similar steps to the manager's maximization problem when he was not large (i.e., in the main sections of the paper) we find:

$$\phi^*=\left(\frac{\mu_F-\gamma\sigma_F\sigma_B\xi}{R_f}\right)\left[\frac{\sigma_F^2\xi+cR_f}{2\sigma_F^2\xi+cR_f}\right]>0$$

□

Proof of Proposition 7: The manager's mean return is now:

$$\mu_F=a-b\theta_F$$

Using this in conumer's problem gives the following maximization problem:

$$\max_{\theta_F,\pi}\left\{\theta_F(a-b\theta_F-\phi R_f)+\pi(\mu_B-R_f)-\frac{\xi}{2}\theta_F^2\sigma_F^2-\frac{\xi}{2}\pi^2\sigma_B^2-\xi\theta_F\pi\gamma\sigma_B\sigma_F\right\}$$

Following the exact same steps from Proposition 1 we find:

$$\theta_F^*=\frac{\left(a-\phi R_f\right)\sigma_B-\left(\mu_B-R_f\right)\sigma_F\gamma}{\sigma_B\left[b+\xi\sigma_F^2\left(1-\gamma^2\right)\right]}$$
$$\pi^*=1$$

The market clearing condition requires the following equality to hold:

$$\mu_B=R_f+\left(\frac{\left(a-\phi R_f\right)\xi\sigma_B\sigma_F\gamma+\xi\sigma_B^2\left[b+\xi\sigma_F^2\left(1-\gamma^2\right)\right]}{b+2\xi\sigma_F^2\left(1-\gamma^2\right)}\right)$$

Once again, we only seek parameter values where the allocations are non-negative. □

Proof of Proposition 8: We proceed as in Proposition 2, however, the optimal value of θ_F chosen by investors is given in Proposition 7 now. Solving:

$$\max_{\phi} \theta_F\left(\phi, \widehat{\phi}\right)\phi - \frac{c}{2}\theta_F^2\left(\phi, \widehat{\phi}\right)$$

yields:

$$\phi^* = \frac{a\left(b + \left(2 - 3\gamma^2\right)\sigma_F^2\xi\right) + \gamma\sigma_F\sigma_B\xi\left(-b + \left(-1 + \gamma^2\right)\sigma_F^2\xi\right)}{2bR_f + \left(4 - 5\gamma^2\right)R_f\sigma_F^2\xi}$$

□