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Pay-to-bid auctions: To bid or not to bid

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ABSTRACT

We propose an all-pay auction format with risk-loving bidders paying a constant fee each time they bid for an object whose monetary value is common knowledge among the bidders, and bidding fees are the only source of benefit for the seller. We characterize a unique symmetric sub-game perfect equilibrium, and further show that the expected revenue of the seller is independent of the number of bidders, decreasing in the sale price and bidding fee, and increasing in the value of the object.

1. Introduction

Over the past few years, certain internet auctions, commonly referred to as pay-to-bid auctions have seen a rapid rise in popularity. The basic structure of the auction is as follows:

- The price starts at zero.
- Each bid costs ¢1 for any bidder.
- Each bid adds 10 s to the game clock so that the auction never ends while there are still willing bidders.
- If the clock reaches zero, the final bidder wins the object and pays the sell price.

When the price is low, each player prefers to bid if afterwards no opponent will enter before the clock runs out. However, if players are too likely to enter in the future, no player will want to bid.

The mechanics of the pay-to-bid auction (also known as penny auction) has parallels with the all-pay auctions and the war-of-attrition game format [3,4]. As with the classic all-pay auction [8], all bidders pay a positive sum if they choose to participate in the auction, regardless of whether they win or lose. The costly action required to become the winning bidder is also reminiscent of the war-of-attrition game format.

Since the main cost incurred by bidders come in the form of bidding fees (which are individually small) in the pay-to-bid auction, bidders are not required to place a bid every round in order to stay in the auction, i.e., entry and re-entry at any round of the auction is allowed whereas re-entry in all-pay auctions is forbidden. Therefore, all-pay auctions never allow the actual winner to pay less than the losers, but in pay-to-bid auctions it happens in practice relatively often.

What really sets this particular auction format apart from other nontraditional auctions is the success of its real world implementation, which appears to be highly profitable for the website operator/auctioneer (who is the sole seller of goods). In December 2008, 14 new websites conducted such auctions; by November 2009, the format had proliferated to 35 websites. Over the same time span, traffic among these sites has increased from 1.2 million to 3.0 million unique visitors per month. For comparison, traffic at ebay.com fluctuated around 75 million unique visitors per month throughout that period. Pay-to-bid auctions have garnered 4% of the traffic held by the undisputed leader in online auctions, [7]. Swoopo.com, is one of such websites that was initially very successful (on 8 February 2012, DealDash the longest running pay-to-bid auction website in the United States acquired the domain Swoopo.com and the URL currently redirects to DealDash's own website.). According to an August 2009 article from *The Economist*, Swoopo has 2.5 million registered users and earned 32 million dollars in revenue in 2008. Data collected by a blogger shows that over April and May of 2009 Swoopo sold items for an average of 188% of their listed value.

There has been a great deal of recent interest in pay-to-bid auctions. Most of this work offers the large profits earned by websites like Sweepo.com (see [1,2,5]), with [7] also focusing on risk loving bidders. This paper differs fundamentally from this work in that its emphasis is on understanding the features of the auction format when bidders are risk loving rather than on explaining empirical observation about bidders' behavior. With this goal in mind, and the extreme success of the internet auctions serving as a motivation, in this paper, we would like to characterize basic properties of a dynamic all pay auction.

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1.1. Risk-loving behavior and having common valuation

These internet auctions are really a form of gambling. The bidder deposits a small fee to play, aspiring to a big payoff of obtaining the item well below its value, with a major difference that the probabilities of winning are endogenously determined. Due to the gambling nature of these auctions and the fact that these internet auctions, like Dealdash.com, prominently advertise themselves as "entertainment shopping", then the risk-loving behavior (i.e., having a preference for risk) by participants of these auctions is natural. For example, for video game systems (such as the Playstation, Wii, or Xbox) the bidders may not be able to justify (to their spouse, their parent, or themselves) spending \$450 on a Xbox at a retail store; however, the potential to win the Xbox early in the auction for only a fraction of that makes it worth the ¢1 gamble, even at unfair odds.

We assume that the valuation of the item is known and the same across all potential bidders. This assumption, unlike in the first and second-price auctions, is quite common in all-pay or war-of-attrition auctions, which are the closest relatives of the pay-to-bid auction. Furthermore, a common value is quite plausible for the types of items regularly auctioned on these internet auctions. All items are new, unopened, and readily available from internet retailers. Hence, the market prices of these items are well established. Indeed, one could consider the value of the object as the lowest price for which the item may readily be obtained elsewhere.

2. Model

Consider the following dynamic game with complete information involving an object with monetary value v and n strategic bidders/players. In each round $t, t \in \{1, 2, 3, \dots\}$, of the auction/game player i chooses an action from the set of pure strategies/messages $S = \{\text{Bid}, \text{No Bid}\}$, $s_{i,t} \in S$, and observes her opponents actions, $s_{j,t}, j \neq i$. In each round of the game, players submit their messages simultaneously. Playing {Bid} is costly. Each player immediately pays $c < v$ dollars (the bid fee) to the seller each time he plays {Bid}. If a player is the sole player who plays {Bid} in any particular round, he wins the object and pays the sale price. The sale price is fixed and denoted by s. In any round of the game, if more than one player plays {Bid} the game continues in the next round. There may arise a difficulty that no participant plays {Bid} in a particular round of the game, then we resolve it by the following assumption.

Assumption 1. If in a round of the game *all* the players play {No Bid}, they resubmit their messages, i.e., they re-play that period of the game.

3. Properties of the game

In this section we investigate the properties of the model proposed in Section 2. We first define the utility function of the players.

3.1. Constant absolute risk-loving utility function

A Von Neumann–Morgenstern utility function $u : \mathbb{R} \to \mathbb{R}$ is said to be Constant Absolute Risk Loving (CARL) if the Arrow–Pratt measure of risk

$$R(x) = -\frac{u''(x)}{u'(x)},$$

is equal to some constant ρ ($\rho < 0$, called the risk-loving coefficient) for all x. Thus, any CARL utility function has the *unique* following form (up to an affine transformation)

$$u(x) = \frac{1 - e^{-\rho x}}{\rho} + K. \tag{1}$$

We consider players are risk-loving and therefore their utility functions are in the form of (1). For simplicity, we focus on the fundamental form in (1) so that $u(0) = 0$, i.e., $K = 0$. Further, we assume players do not discount future consumption.

Remark 1. In the limit when ρ tends to 0 the utility function in (1) converges (point wise) to the risk neutral utility function $u(x) = x$, for all x.

3.2. Equilibrium analysis

In the following theorem, we establish that for the model proposed in Section 2 there exists a *unique* Symmetric Sub-game Perfect Equilibrium (SSPE) which is *stationary*. We note that, here, we focus on characterizing symmetric equilibria. There might be asymmetric equilibria but they are not of our interest. In Corollary 1 we comment on the appropriateness of analyzing symmetric equilibria.

Theorem 1. *In any SSPE of the game proposed in Section 2, with $n \geq 2$ players, the following properties are satisfied.*

I. *In any sub-game starting from round $t, t \in \{1, 2, 3, \dots\}$, the expected utility of player i whose wealth/budget level is $w_{i,t}$, is exactly equal to $u(w_{i,t})$.*
II. *In each period, each player* purely *randomizes over* {Bid, No Bid} *and chooses to play* {Bid} *with the following* stationary *probability*

$$1 - \sqrt[n-1]{\frac{u(c)}{u(v-s)}},$$

and the SSPE is unique.

Proof. See Appendix. □

From the above theorem we directly obtain the following set of corollaries for the specified SSPE.

Corollary 1. *Since the decision taken by the players is* independent *of their wealth level, we are able to focus on the* symmetric *sub-game perfect equilibria despite wealth asymmetries.*

Corollary 2. *In each period of the game the probability of playing* {Bid} *is* strictly *greater than zero and less than one. That is, in each period of the game each player is* indifferent *between playing* {Bid} *and* {No Bid}.

Corollary 3. *Theorem 1 is also satisfied when players are risk-neutral (the risk neutral case can be easily implied by taking a limit $\rho \to 0$ in (1), that implies (point-wise) $u(x) = x$ for any x). In particular, when players are risk neutral, in any round $t, t \in \{1, 2, 3, \dots\}$, of the game each player chooses to play* {Bid} *with the stationary probability* $1 - \sqrt[n-1]{\frac{c}{v-s}}$.

Corollary 4. *The probability that any bidding player, i.e., a player who plays* {Bid}*, wins the object in any round of the game is equal to* $\frac{u(c)}{u(v-s)}$.

3.3. Revenue analysis

In Theorem 2 we explicitly characterize the expected revenue of the seller. In particular, we show that the expected revenue of the seller, is decreasing in the sale price (s), increasing in the value of the object (v) and, decreasing in the bidding fee (c). Further, we see that the expected revenue of the seller is independent of the number of players. This results are intuitive, because risk-loving bidders prefer low probability and potentially very lucrative gambles to gamble that are equal in expectations but have lower variance. The seller is able to earn large profits even when the bid fee and sale price are small compared to the object's value because the expected number of bids and the expected length of these auctions are large enough to compensate this discrepancy.

Theorem 2. *In the SSPE of the Bid-No Bid game, if players are risk neutral (i.e., $\rho = 0$ in (1)), then*

I. *The expected revenue of the seller is equal to the value of the object (v).*

if players are risk-loving (i.e., $\rho < 0$ in (1)), then

II. *The expected revenue of the seller is equal to $\frac{u(v-s)c}{u(c)} + s$.*
III. *The expected revenue of the seller is independent of the number of players.*
IV. *The expected revenue of the seller is strictly greater than the value of the object (v).*
V. *The expected revenue of the seller is strictly increasing in the value of the object (v), decreasing in the sale price (s), decreasing in the bidding fee (c), and decreasing in ρ (i.e., the seller earns more when players are more risk-loving).*
VI. *The maximum revenue the seller can earn is $u(v) = \frac{1-e^{-\rho v}}{\rho}$.*

Proof. See Appendix. □

3.4. A standard lottery with risk-loving players

Finally, in this section we show for any number of players, for a sufficiently small bidding fee, the seller's revenue in the game proposed in Section 2 is strictly *greater* than a standard lottery.

Consider a seller offering lottery tickets for a prize/object of (monetary) value v to n potential buyers with CARL utility. The cost of buying a ticket for each player is equal to $c > 0$, and the winner is chosen randomly. Thus, the maximal price the seller can choose to charge for a (lottery) ticket is the solution of $u(c) = \frac{1}{n}u(v)$. Hence, the optimal (revenue maximizing) ticket price is $c^* = u^{-1}\left(\frac{1}{n}u(v)\right)$. Since for all $x > 0$, $u(x) > x$, thus for all $x > 0$, $x > u^{-1}(x)$. Therefore, $\frac{1}{n}u(v) > u^{-1}\left(\frac{1}{n}u(v)\right)$, and consequently for all $n \in \mathbb{N}$, $u(v) > n \times u^{-1}\left(\frac{1}{n}u(v)\right) = nc^* =$ the maximum seller's revenue from the lottery.

Note that by part VI. of Theorem 2, $u(v)$ is the maximum seller's revenue in the game proposed in Section 2. Thus, the above inequality shows that, for a sufficiently small bidding fee, the seller's revenue in the game proposed in Section 2 is strictly greater than a standard lottery, for any number of players.

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Appendix

Notation. The history of the game up to time t, $t \in \{1, 2, 3, \ldots\}$, which is common knowledge among the players, is denoted by $h^t = (\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{t-1})$, where $\mathbf{s}_k = (s_{1,k}, s_{2,k}, \ldots, s_{n,k})$ represents the strategy profile reported by the players in round k of the game. Consider a sub-game beginning after history h^t. $\mathbb{E}_{i,t}[u(X_{h^t})]$ is the expected utility of player i in this sub-game and, $\mathbb{E}_{i,t}[u(X_{h^t})|\{\text{Bid}\}]$ ($\mathbb{E}_{i,t}[u(X_{h^t})|\{\text{No Bid}\}]$) is the expected utility of player i in this sub-game given that he plays {Bid} ({No Bid}) in round t and, X_{h^t} is the random variable denoting money earned by the player in equilibrium in the sub-game starting after h^t. Each player at any round t chooses a strategy which is a map from any history of the game up to time t to [0, 1], a Bernoulli probability over {Bid, No Bid}. Let $p_t(h^t)$, $0 \le p_t(h^t) \le 1$, be the probability of choosing {Bid} after observing history h^t of the game.

Proof of Theorem 1. We prove each part of the theorem, separately, as follows.

Proof of I.: We note that in any SSPE, at any round t, $0 < p_t(h^t) < 1$, because, due to the game specification, for example if $p_t(h^t) = 1$ then for any player it is profitable to unilaterally deviate to {No Bid} (a similar argument holds when $p_t(h^t) = 0$). Further, it can be shown that there exists δ such that at any round t of the game $p_t(h^t) > \delta$. We prove this by contradiction. Suppose that there is not such δ, i.e., for any $\delta > 0$ there exists history h^t, which occurs with positive probability in equilibrium, such that $p_t(h^t) < \delta$. Thus, by picking δ sufficiently small we have:

$$\begin{aligned}\mathbb{E}_{i,t}[u(X_{h^t})|\{\text{Bid}\}] &\overset{(a)}{>} (1-\delta)^{n-1}u(v-s+w_i-c)\\ &\ge \frac{u(v)}{n}\\ &\ge \mathbb{E}_{i,t}[u(X_{h^t})]\\ &\ge \mathbb{E}_{i,t}[u(X_{h^t})|\{\text{No Bid}\}], \end{aligned} \tag{2}$$

where (a) follows since δ is very small. Next, we show that in any sub-game perfect equilibrium starting in round t of the game each player earns expected utility $u(w)$ where w is the wealth of the player in round t. First, we show this statement is true when $w = 0$, that is, $\mathbb{E}_{i,t}[u(X_{h^t})] = 0$ when $w = 0$ in round t. We prove it by contradiction. Suppose that there exists history h^t, which occurs with positive probability in equilibrium, such that

$$\mathbb{E}_{i,t}[u(X_{h^t})] = \gamma > 0. \tag{3}$$

As we proved in the above, there exists δ such that $p_t(h^t) > \delta$, $\forall\, t$. Further, since, $p_t(h^t) \in (0, 1)$, $\forall t$, thus each player should be indifferent between choosing {Bid} and {No Bid}, in each round t. Also, since $\mathbb{E}_{i,t}[u(X_{h^t})] = \gamma > 0$, there exists h^s including h^t, i.e., $h^t \subset h^s$, such that $\mathbb{E}_{i,t}[u(X_{h^s})] \ge \gamma$. Thus, due to the fact that player i is indifferent between {Bid} and {No Bid} we have:

$$\begin{aligned}\mathbb{E}_{i,s}[u(X_{h^s})] &= \mathbb{E}_{i,s}[u(X_{h^s})|\{\text{No Bid}\}]\\ &= \Lambda_{i,s}\ \mathbb{E}_{i,s+1}[u(X_{h^{s+1}})] \ge \gamma\\ &\Rightarrow \mathbb{E}_{i,s}[u(X_{h^{s+1}})] \ge \frac{\gamma}{\Lambda_{i,s}}, \end{aligned} \tag{4}$$

where, $\Lambda_{i,s}$ is the probability that at least two players (except i) or none of them play {Bid} in round s, given h^s, i.e.,

$$\Lambda_{i,s} = 1 - \binom{n-1}{1} p_s(h^s)(1-p_s(h^s))^{n-2}.$$

Furthermore, note that $0 < \Lambda_{i,s} < 1$. Using the new lower bound derived in (4) and, following similar arguments as we did to derive (4) imply

$$\mathbb{E}_{i,s+2}[u(X_{h^{s+2}})] \ge \frac{\gamma}{\Lambda_{i,s}\Lambda_{i,s+1}}, \quad h^{s+2} \subset h^{s+1} \subset h^s. \tag{5}$$

Following the above arguments we obtain

$$\mathbb{E}_{i,s+r}[u(X_{h^{s+r}})] \ge \frac{\gamma}{\prod_{k=0}^{r-1} \Lambda_{i,s+k}}, \tag{6}$$

where $h^{s+r} \subset \cdots \subset h^{s+1} \subset h^s$. But notice that since for any $k \geq 0$, $0 < \Lambda_{i,s+k} < 1$, thus by choosing r sufficiently large we can get

$$\frac{\gamma}{\prod_{k=0}^{r-1} \Lambda_{i,s+k}} > u(v), \tag{7}$$

which is a contradiction, because $\mathbb{E}_{i,s+r}[u(X_{h^{s+r}})]$ cannot exceed $u(v)$. Therefore,

$$\mathbb{E}_{i,t}[u(X_{h^t})] = 0. \tag{8}$$

Now, suppose that $w > 0$, therefore Eq. (8) along with (1) imply

$$\begin{aligned}
\mathbb{E}_{i,t}[u(w + X_{h^t})] &= \mathbb{E}_{i,t}\left[\frac{1 - e^{-\rho(w+X_{h^t})}}{\rho}\right] \\
&= \frac{1 - \mathbb{E}_{i,t}\left[e^{-\rho(w+X_{h^t})}\right]}{\rho} \\
&= \frac{1 - e^{-\rho w}\mathbb{E}_{i,t}\left[e^{-\rho X_{h^t}}\right]}{\rho} \\
&= \frac{1 - e^{-\rho w} + e^{-\rho w}\left(1 - \mathbb{E}_{i,t}\left[e^{-\rho X_{h^t}}\right]\right)}{\rho} \\
&= \frac{1 - e^{-\rho w}}{\rho} + e^{-\rho w}\mathbb{E}_{i,t}\left[\frac{1 - e^{-\rho X_{h^t}}}{\rho}\right] \\
&= \frac{1 - e^{-\rho w}}{\rho} + e^{-\rho w}\mathbb{E}_{i,t}[u(X_{h^t})] \\
&= u(w).
\end{aligned} \tag{9}$$

The above equality completes the proof of the first part of the theorem.

Before proving the next part, we present the following remark that holds in risk-loving utility functions.

Remark 2 (*See [6]*)**.** Let $u(\cdot)$ be a CARL utility function. Suppose X and Y are two random variables and α is a constant. Then,

$$\mathbb{E}[u(X + \alpha)] \geq \mathbb{E}[u(Y + \alpha)] \iff \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)].$$

Remark 2 states that comparing two random variables is independent of a constant shift when the utility function has a CARL form, like what we defined in (1).

Proof of II.: The expected utility of player i if he plays {Bid} (with wealth level equals to $w_{i,t}$ at the beginning of round t) in equilibrium in the sub-game starting after history h^t is equal to,

$$\begin{aligned}
&\mathbb{E}_{i,t}[u(w_{i,t} + X_{h^t})|\{\text{Bid}\}] \\
&\quad = u(w_{i,t} + v - s - c)(1 - p_t(h^t))^{n-1} \\
&\qquad + \sum_{k=1}^{n-1}\left[\binom{n-1}{k} p_t(h^t)^k(1 - p_t(h^t))^{n-1-k}\right. \\
&\qquad \left. \times \mathbb{E}_{i,t+1}[u(w_{i,t} - c + X_{h^t \cup \mathbf{s}^k_{t+1}})]\right],
\end{aligned} \tag{10}$$

where $\mathbf{s}^k_{t+1}$ is the strategy profile in round $t + 1$ that k players play {Bid} and $h^t \cup \mathbf{s}^k_{t+1}$ is the updated history. In (10), the first term corresponds to the event that player i wins the object, i.e., the event in which player i is the only player who plays {Bid} in round t and, the second term corresponds to the expected utility of player i from continuing after history h^t, equivalently, more than one player plays {Bid}.

Similarly, if player i plays {No Bid}, then

$$\begin{aligned}
&\mathbb{E}_{i,t}[u(w_{i,t} + X_{h^t})|\{\text{No Bid}\}] \\
&\quad = u(w_{i,t})\binom{n-1}{1}(1 - p_t(h^t))^{n-2}p_t(h^t) \\
&\qquad + \sum_{k\neq 1}\left[\binom{n-1}{k} p_t(h^t)^k(1 - p_t(h^t))^{n-1-k}\right. \\
&\qquad \left. \times \mathbb{E}_{i,t+1}[u(w_{i,t} + X_{h^t \cup \mathbf{s}^k_{t+1}})]\right].
\end{aligned} \tag{11}$$

In (11), the first term corresponds to the event that player j ($j \neq i$) wins the object, i.e., the event in which player j ($j \neq i$) is the only player who plays {Bid} in round t and, the second term corresponds to the expected utility of player i from continuing after history h^t, equivalently, more than one player other than i plays {Bid}.

Since player i purely randomizes between {Bid} and {No Bid}, then player i is indifferent between choosing {Bid} and {No Bid}, that is, (10) is equal to (11).

Now, using Remark 2 enables us to simplify (10) and (11) as follows. Without loss of generality, we can set $w_{i,t} = c$ and, consequently, obtain

$$\begin{aligned}
\mathbb{E}_{i,t+1}[u(w_{i,t} - c + X_{h^t \cup \mathbf{s}^k_{t+1}})]|_{w_{i,t}=c} &= \mathbb{E}_{i,t+1}[u(X_{h^t \cup \mathbf{s}^k_{t+1}})] \\
&\overset{(a)}{=} 0,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\mathbb{E}_{i,t+1}[u(w_{i,t} + X_{h^t \cup \mathbf{s}^k_{t+1}})]|_{w_i=c} &= \mathbb{E}_{i,t+1}[u(c + X_{h^t \cup \mathbf{s}^k_{t+1}})] \\
&\overset{(b)}{=} u(c),
\end{aligned} \tag{13}$$

where (a) follows from (8), and (b) from (9).

Plugging $w_i = c$ into (10) (because of Remark 2) and using (12), Eq. (10) can be simplified as follows,

$$\mathbb{E}_{i,t}[u(w_{i,t} + X_{h^t})|\{\text{Bid}\}]|_{w_{i,t}=c} = u(v - s)(1 - p_t(h^t))^{n-1}, \tag{14}$$

and similarly, plugging $w_{i,t} = c$ into (11) and employing (13), Eq. (11) is simplified as follows

$$\begin{aligned}
&\mathbb{E}_{i,t}[u(w_{i,t} + X_{h^t})|\{\text{No Bid}\}]|_{w_{i,t}=c} \\
&\quad = u(c)\binom{n-1}{1}(1 - p_t(h^t))^{n-2}p_t(h^t) \\
&\qquad + \sum_{k\neq 1}^{n-1}\binom{n-1}{k} p_t(h^t)^k(1 - p_t(h^t))^{n-1-k}u(c) \\
&\quad = u(c)\left[\binom{n-1}{1}(1 - p_t(h^t))^{n-2}p_t(h^t)\right. \\
&\qquad \left. + \sum_{k\neq 1}^{n-1}\binom{n-1}{k} p_t(h^t)^k(1 - p_t(h^t))^{n-1-k}\right] \\
&\quad = u(c).
\end{aligned} \tag{15}$$

Finally, since player i is indifferent between choosing {Bid} and {No Bid}, thus equating (14) and (15) gives that

$$p_t(h^t) = 1 - \sqrt[n-1]{\frac{u(c)}{u(v - s)}}. \tag{16}$$

It is immediate from (16) that the characterized symmetric equilibrium is unique and stationary since it is independent of time and controlled by the (n, v, s, c). □

Proof of Theorem 2. The probability of the event that the game ends in round t given that t is reached is equal to

$$h_t = \frac{np_t(1-p_t)^{n-1}}{1-(1-p_t)^n}, \tag{17}$$

where h_t is called *hazard rate* at time t.

The probability that a round t of the game is reached, is equal to the probability that the game does not end in any round s, $s < t$, which is equal to

$$\prod_{s=1}^{t-1}(1-h_s) = \prod_{s=1}^{t-1}\left(1 - \frac{np_s(1-p_s)^{n-1}}{1-(1-p_s)^n}\right). \tag{18}$$

Now, let Q_t denote the expected number of entrants in round t of the game. Thus,

$$\begin{aligned} Q_t &= \sum_{k=1}^{n}\binom{n}{k} p_t^k(1-p_t)^{n-k}k + (1-p_t)^n Q_t \\ &= \sum_{k=0}^{n}\binom{n}{k} p_t^k(1-p_t)^{n-k}k + (1-p_t)^n Q_t \\ &= np_t + (1-p_t)^n Q_t. \end{aligned} \tag{19}$$

Eq. (19) implies that,

$$Q_t = \frac{np_t}{1-(1-p_t)^n}. \tag{20}$$

The seller's expected earnings from the bid fees in round t is the expected number of entrants times the bid fee times the probability that round t is reached, that is:

$$Q_t \times c \times \prod_{s=1}^{t-1}(1-h_s) = \frac{np_t}{1-(1-p_t)^n} \times c \times \prod_{s=1}^{t-1}\left(1 - \frac{np_s(1-p_s)^{n-1}}{1-(1-p_s)^n}\right).$$

Therefore, the seller's expected earning from the bid fees throughout the game is exactly equal to

$$\sum_{t=1}^{\infty}\left[Q_t \times c \times \prod_{s=1}^{t-1}(1-h_s)\right] = \sum_{t=1}^{\infty}\left[\frac{np_t}{1-(1-p_t)^n} \times c \times \prod_{s=1}^{t-1}\left(1 - \frac{np_s(1-p_s)^{n-1}}{1-(1-p_s)^n}\right)\right]. \tag{21}$$

As we proved in Theorem 1, in equilibrium, in each round of the game, each player chooses to play {Bid} with the following stationary probability

$$p_t = p = 1 - \sqrt[n-1]{\frac{u(c)}{u(v-s)}}. \tag{22}$$

Since the seller's expected revenue is equal to the sale price (denoted by s) plus the seller's expected earning from the bid fees throughout the game, thus, by plugging (22) into (21), we obtain

Seller's expected revenue

$$\begin{aligned} &= s + \sum_{t=1}^{\infty}\left[\left(c\frac{np}{1-(1-p)^n}\right)\left[1 - \frac{np(1-p)^{n-1}}{1-(1-p)^n}\right]^{t-1}\right] \\ &= s + \left(c\frac{np}{1-(1-p)^n}\right)\sum_{t=1}^{\infty}\left[1 - \frac{np(1-p)^{n-1}}{1-(1-p)^n}\right]^{t-1} \\ &= s + \left(c\frac{np}{1-(1-p)^n}\right)\frac{1-(1-p)^n}{np(1-p)^{n-1}} \\ &= s + \frac{c}{(1-p)^{n-1}} \\ &\overset{(a)}{=} s + c\frac{u(v-s)}{u(c)}. \end{aligned} \tag{23}$$

Proof of I.: When players are risk neutral (since $u(x) = x$), Eq. (23) implies that

$$\text{Seller's expected revenue (with risk neutral utility)} = v. \tag{24}$$

Proof of III.: It is immediate from (23) that the seller's expected revenue is independent of the number of players.

Proof of IV.: We show when players are risk-loving, i.e., $\rho \neq 0$,

$$s + c\frac{u(v-s)}{u(c)} > v. \tag{25}$$

To prove (25) we use the following lemma.

Lemma 3. *Let $u(\cdot)$ be a convex function and $x > 0$ is a constant. Define*

$$f(\alpha) := u((1+\alpha)x) - (1+\alpha)u(x).$$

Then, $f(\alpha) > 0$, where $\alpha > 0$.

Proof of Lemma 3. To prove Lemma 3, we first show that

$$\begin{aligned} \frac{u(x)}{x} &= \frac{\int_0^x u'(t)dt}{x} \overset{(a)}{<} \frac{\int_0^x u'(x)dt}{x} = u'(x) \\ &\Rightarrow u(x) < xu'(x), \end{aligned} \tag{26}$$

where (a) follows because $u'(\cdot)$ is increasing. Using (26) we obtain

$$f'(\alpha) = xu'((1+\alpha)x) - u(x) > xu'(x) - u(x) > 0. \tag{27}$$

Thus $f(\alpha)$ is increasing in α and $f(\alpha) > f(0) = 0$. □

A direct consequence of Lemma 3 is the following

$$x > y > 0 \Rightarrow \frac{u(x)}{u(y)} > \frac{x}{y}. \tag{28}$$

The above relation follows by setting $x = (1+\alpha)y$, where $\alpha > 0$, and Lemma 3.

Now, we can prove that (25) holds, because

$$s + c\frac{u(v-s)}{u(c)} - v = c\left[\frac{u(v-s)}{u(c)} - \frac{v-s}{c}\right] \overset{(a)}{>} 0, \tag{29}$$

where (a) follows because $v - s > c$ and (28).

Proof of V.: In the following, we show that the seller's revenue is increasing in v, decreasing in s and decreasing in c.

$$\frac{\partial\left[s + c\frac{u(v-s)}{u(c)}\right]}{\partial v} > 0,$$

$$\frac{\partial\left[s + c\frac{u(v-s)}{u(c)}\right]}{\partial s} = 1 - c\frac{u'(v-s)}{u(c)} < 1 - c\frac{u'(c)}{u(c)} \overset{(a)}{<} 0, \tag{30}$$

$$\frac{\partial\left[s + c\frac{u(v-s)}{u(c)}\right]}{\partial c} = \frac{u(v-s)u(c) - cu'(c)u(v-s)}{u(c)^2} = \frac{u(v-s)}{u(c)^2}(u(c) - cu'(c)) \overset{(b)}{<} 0,$$

where (a) and (b) are correct because of (26).

Proof of VI.: Because of the results of the previous part, the expected revenue of the seller is decreasing in s and decreasing in c. Therefore, in order to find the maximum value of the seller's expected revenue (when v is given), in (23) we set $s = 0$ and take a limit when $c \to 0$ as follows:

$$\max_{s,c}\left[s + c\frac{u(v-s)}{u(c)}\right] = \lim_{c\to 0}\frac{cu(v)}{u(c)} = \frac{u(v)}{u'(0)} = \frac{1 - e^{-\rho v}}{\rho} = u(v). \quad \square \tag{31}$$

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