Multistage Financing: Milestone Bonuses or Deferred Compensation*

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Abstract

We develop a dynamic principal-agent model for financing a multistage project. The optimal contract displays the following unique features: (i) There is a *pecking order* between milestone bonuses and deferred compensation: when an intermediate stage succeeds, principal prefers to use deferred compensation as much as possible to reward agent before using milestone bonuses; and (ii) the agent's equity stake in the project is positively related to performance and is also smaller in later stages than in early ones. Moreover, when the principal observes a signal about the agent's action, the incentive provided by monitoring and reward dynamically *complement* each other: When negative (positive) evidence about the agent's effort accumulates, the principal increases (decreases) contract sensitivity on the signal, monitors the agent more closely (loosely), and reduces (increases) the reward for success. We relate our model predictions to the empirical literature on venture capital financing and provide new empirical tests.

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Introduction

Research and development (R&D) is an important contributor to economic growth. As Grossman and Helpman (1994) note, technological innovation has been the driving force behind inexorably rising standards of living over the long run. One important form of corporate innovation is high-growth startups backed by venture capital (VC). These startups play a disproportionate role in innovation and productivity growth (Kortum and Lerner (2000); Puri and Zarutskie (2012)).

R&D projects and VC-backed startups share several common features. First, both are typically organized under an agency relationship that combines both technical expertise and capital. This form of separation between technical experts and financiers leads to misalignment. In addition, financiers cannot fully monitor or understand actions taken by the technical experts who are responsible for executing the project or venture. This creates moral hazard between technical experts and financiers. Second, complex projects often require the completion of multiple sequential stages before their final benefits can be realized. For example, new drug development requires multiple steps: laboratory testing, animal trials, and human trials. Only after all these sequential stages are successfully completed can the approval request be submitted. Similarly, VC-backed startups typically raise funding in multiple stages (Gompers (1995)). Third, R&D projects and startups heavily depend on a small group of key personnel. One important source of the extreme skewness in startups' performance is due to the initial human capital endowment of their founding entrepreneur (Lazear (2004)). As an example, after Steven Jobs was forced out of Apple in 1985, the company struggled due to competition from Microsoft and IBM. Jobs returned to Apple in 1997 as the CEO. He was largely responsible for helping revive Apple, which had been on the verge of bankruptcy. The NeXTSTEP platform, developed by Jobs' company NeXT during his years away from Apple, became the foundation Apple's new operating system Mac OS X. This example clearly shows that Steven Jobs' human capital is vital for Apple's early success.

Motivated by these three salient features of a complex project, we consider a dynamic principal-agent model in which a risk-neutral principal (she) offers a contract to a risk-neutral agent (he) in order to execute a project. The project has multiple stages. Later stages start only when the previous stages are successfully completed. The project does not provide intermediate cash flow before the project completion but generates a lump-sum payoff when the final stage succeeds. Due to uncertainties and challenges, completion of different stages arrive randomly, which we model using the jump time of an inhomogeneous Poisson process. The agent's effort determines the intensity of the Poisson process: high (low) effort increases (decreases) the jump intensity, which reduces (prolongs) the expected completion time of each stage. The agent's effort also positively impacts the probability of success at each stage. If the agent shirks from the recommended effort, he receives a flowing private benefit. The principal cannot monitor the agent's effort directly, creating moral hazard in the project.

The agent is protected by limited liability. When the agent's value from the project reaches an outside value, reflecting the agent's market worthiness, the agent quits to protect his limited liability, and the project is terminated before completion. Project termination is inefficient for the principal because the project is prematurely stopped before realizing its potential. This condition reflects the importance of the agent's human capital to the project.

In our baseline model, the principal uses a sequence of deferred compensation and milestone cash bonuses at the end of each stage to incentivize the agent. When an intermediate stage is successfully completed, the principal can either reward the agent with milestone cash bonuses or deferred compensation, which promises the agent future rewards if further milestones are achieved. We show that there is a *pecking order* among these two forms of compensation: The principal prefers deferred compensation to milestone bonuses. In fact, only when the deferred compensation reaches its maximum level does the principal award the agent with milestone bonuses. The reason for this result is that the marginal benefit of rewarding the agent with cash is less than the marginal cost, due to the possibility of inefficient project termination. The principal prefers to use deferred compensation to maintain the agent's skin in the project and mitigate the risk of inefficient project termination. Optimal milestone bonuses are awarded using a threshold strategy only when the agent's continuation utility is sufficiently high, such that the marginal benefit of cash bonuses equals the marginal cost. In that case, the principal tops up deferred compensation with milestone bonuses.

This result is related to the evolution of founder-CEO compensation in VC-backed startups studied by Ewens, Nanda, and Stanton (2020). The authors show that having a "product–market fit" is a significant milestone, after which cash compensation increases significantly for founder-CEOs. They argue that product–market fit reduces the reliance on key human capital and reduces the threat of inefficient project termination due to the departure of key personnel. This evidence is consistent with our result in that only when the project is sufficiently far from termination is the agent compensated with cash.

Our model also shows that the agent's equity stake is smaller in later stages than in early stages. This is due to the resolution of uncertainty in later stages, which increases the project value for the principal. For the same agent's continuation utility, the agent's proportion in the value of the project and his equity stake decrease as a result. A project faces various sources of uncertainty. If early stages fail, the project cannot reach its final completion to realize its payoff. In addition, the project payoff can be uncertain because a low type project generates a low payoff even if all stages are successful. In later stages, project success is more certain. Because early stages have been successfully completed, the principal also learns about the viability of the project and terminates low type projects in early stages. Both circumstances lead to higher expected payoffs in later stages and, therefore, higher project values for the principal. This result is consistent with Bengtsson and Sensoy (2015), who document that an entrepreneur's equity stake is lower in later funding rounds. In addition, Ewens, Nanda, and Stanton (2020) show that an entrepreneur's equity stake decreases with firm age.

We consider a natural extension of the baseline model in which the principal can monitor the agent's effort by observing a signal about the agent's effort. The signal can be interpreted as either the agent's working hours or as regularly submitted status updates, both of which are noisy proxies for the agent's true effort. In this model extension, the principal can improve the signal accuracy, but subject to a flow cost proportional to the signal precision. A negative signal is interpreted by the principal as the agent shirking from his recommended effort, and the agent is punished by the principal when negative signals accumulate. Conversely, the agent is rewarded when positive signals of his effort are observed. Therefore, monitoring can be used as a "stick" that punishes the agent for negative evidence. Meanwhile, deferred compensation and milestone bonuses function like a "carrot" that rewards the agent for successful completion of each stage. The principal combines the carrot and the stick to incentivize the agent.

We show that monitoring and reward *complement* each other in the optimal contract. When positive evidence of the agent's effort accumulates, the principal increases the reward and relaxes monitoring. As a result, the optimal contract relies more on the carrot to motivate the agent. Meanwhile, when negative evidence of the agent's effort accumulates, the principal gradually replaces the carrot with the stick — reducing the reward and increasing the contract sensitivity to the signal, which elevates the stick and penalizes the agent more if further negative evidence continues to build. The intuition for this dynamic and complementary relation is as follows: When negative evidence about the agent's effort accumulates, the agent's continuation utility decreases and the project moves closer to inefficient termination. In this case, a negative component in the agent's expected continuation utility, induced by the reward, further reduces the distance between the agent's continuation utility and his outside value, thereby decreasing the distance to the project's inefficient termination. To mitigate termination risk, the principal gradually reduces her reward and increases the contract sensitivity of monitoring. Even though volatility in the agent's continuation utility also increases with contract sensitivity, the effect of volatility on the agent's continuation utility depends on whether incoming shocks are positive or negative.

This impact on volatility is less damaging than a reduction in the expected continuation utility induced by the reward.

Not only does the reward increase with performance, our model shows that the state-dependence is also more significant in early stages than in later ones. This finding is consistent with the empirical evidence of Kaplan and Strömberg (2003). The authors document that entrepreneurs have more cash flow rights under good performance and that state-contingencies are significantly greater in first VC rounds compared to subsequent ones.

When the project is close to its inefficient termination, the principal not only increases the contract sensitivity of the signal, but also increases the signal precision, thereby monitoring the agent more closely. When the project is far from its termination, the principal loosens monitoring and decreases the signal precision. This result connects to the board supervision in Kaplan and Strömberg (2003). The authors show that VCs get more board control rights and voting control after poor performance. However, as performance improves, VCs return most control rights to the entrepreneurs.

The optimal contract is implemented via long-term debt, equity, and a credit line. The maximum allowance in the credit line varies with the stage. At the beginning of each stage, the balance on the credit line is reduced or cleared to start a new stage.

Beyond the connections to the empirical literature mentioned above, we also propose two empirical tests of our model predictions. To test the pecking order between milestone bonuses and deferred compensation, we propose using the data set from Oyer (2004) and Oyer and Schaefer (2005). They empirically show that firms generally prefer granting deferred compensation (e.g., stock options) to their employees over paying cash bonuses (milestone bonuses) in order to improve retention. It would be interesting to investigate the threshold under which firms prefer to pay via deferred compensation rather than a cash salary. To test the complementarity between monitoring and reward, we propose using the data set from Bernstein, Giroud, and Townsend (2016). They show that the introduction of new airline routes that reduce VCs' travel times to their existing portfolio companies increase VCs' involvement and the likelihood of successful exit. Using VC contracts with their portfolio companies, researchers could examine whether cashflow rights, equity stakes, or other forms of reward are reduced after the introduction of new airline routes.

Related Literature. Our paper is related to the literature on contracting for multistage projects, for example Green and Taylor (2006) and Mayer (2021). In these two papers, intermediate progress or project failure is privately observed. Green and Taylor (2006) find that a soft deadline and self-reporting can mitigate the incentive of shirking as the hard deadline approaches. Mayer (2021) finds that payment for failure after an unconditional financing period incentivizes agents to truthfully report failure. Our model assumes that intermediate progress or failure is publicly observable. Different from the aforementioned two papers, the project in our model is inefficiently terminated when the agent's continuation utility in the project reaches his outside value. This provision reflects the agent's limited commitment in his human capital. We focus on the impact of this agency friction on the choice of deferred compensation versus milestone bonuses at intermediate progress points. Toxvaerd (2006) studies a multistage project with observable completions. The optimal contract takes into account both risk sharing and incentive provision. However, the project in Toxvaerd (2006) does not involve inefficient termination. Varas (2018) examines the impact of managerial short-termism. The optimal contract breaks up into two phases in the state space of agent's continuation utility. When the agent's rent is low, the contract enters a stationary phase, where incentives are provided by the threat of termination.

Our model is also related to the literature on monitoring in contracting settings. Beginning with Townsend (1979), a variety of studies have used monitoring to conduct costly state verification in a static setting with adverse selection issues. Dye (1986) examines costly state verification in a static setting with moral hazard concerns. In continuous-time dynamic settings, Piskorski and Westerfield (2016), Malenko (2019), Varas, Marinovic, and Skrzypacz (2020), and Chen, Sun, and Xiao (2020) analyze optimal dynamic contracts when the principal can monitor the agent's hidden action or the project's hidden type.

Piskorski and Westerfield (2016) predict a hump-shaped relation between monitoring and an agent's continuation utility because of the limited scope of punishment when the agent's continuation utility is low. In our model, the principal chooses the magnitude of the penalty via the contract sensitivity to a monitored signal. In the optimal contract, the agent is penalized more severely when negative evidence against his effort accumulates. This induces a monotone decreasing relation between monitoring and the agent's continuation utility. In particular, when the agent's continuation utility reaches its maximum value, monitoring is turned off completely.

The optimal monitoring in our model resembles the "lower-tailed" structure in Dye (1986): Monitoring is more intense (both choosing high signal precision and imposing a harsh penalty) when negative evidence accumulates. Monitoring also takes place in Chen, Sun, and Xiao (2020) when the agent's continuation utility is low and in Malenko (2019) when the reported project type is high. Our mechanism of the lower-tailed structure is different: Due to the possibility of inefficient project termination, the marginal benefit of monitoring is high when the project is close to termination. Our model also combines the punishment mechanism of monitoring and the reward mechanism of bonuses, showing the complementarity between monitoring and reward. While the optimal monitoring in our model is continuous, the optimal monitoring in Varas, Marinovic, and Skrzypacz (2020) can be either deterministic with periodic reviews or random, depending on whether information or incentive provision is the principal's main concern.

Our paper builds on the literature on dynamic contracting in continuous time, which began with the seminal papers of DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), and Sannikov (2008) and includes more recent contributions by He (2009, 2011, 2012), Biais, Mariotti, Rochet, and Villeneuve (2010), Hoffmann and Pfeil (2010), DeMarzo, Fishman, He, and Wang (2012), Zhu (2013), and Williams (2015), among others. Our paper focuses on the dynamic mechanism in multistage settings and analyzes the interaction between monitoring and reward in the context of multistage project financing. Similar to papers with Poisson arrivals (e.g., Biais, Mariotti, Rochet, and Villeneuve (2010), Hoffmann and Pfeil (2010), Wong (2019)), in our model, Poisson jumps arrive finite times and can lead to potential good news: stage success.

Our paper connects to the literature on learning and optimal incentive schemes for venture capital, such as Bergemann and Hege (1998, 2005). In these two papers, financiers learn the project type via Bayesian updating. In one extension of the baseline model, the principal can exercise a costly real option to investigate the viability of the project. Our paper is also related to the literature on optimal contracting with limited commitment. (See e.g., Ai and Li (2015) and Bolton, Wang, and Yang (2019), among others.)

The rest of the paper is organized as follows. Section 1 presents the model setup. Section 2 presents a baseline model in which a principal uses a sequence of deferred compensation and milestone bonuses to incentivize effort. An extension of the baseline model is introduced in Section 3, where the principal can contract on a signal about the agent's effort as well as provide rewards for success. The optimal contract is implemented in Section 4. Two further extensions are discussed in Section 5. We propose two empirical tests to our model in Section 6. The paper is concluded in Section 7 and all proofs are provided in the Appendix.

1 Model

1.1 Project setting

Our model considers a principal who offers a contract to an agent in order to complete a project. The project has multiple stages. A later stage can only be initiated when the previous stage is successfully completed. The project has a total of N stages. The project does not provide intermediate cash flows before the completion of the final stage, but it generates a lump-sum payoff D if the project is successful at its final stage, or zero payoff if it fails at at any stage.

Due to the high uncertainty of the project, completion of each stage arrives randomly (if ever) and requires the agent's continuous effort. We model the completion time of different stages as the jump time of an inhomogeneous Poisson process whose intensity is determined by the agent's effort. If the agent exerts effort a_t at time t, the current stage completes with probability $a_t dt$ in the next infinitesimal dt time, or the project continues into the next period with probability $1 - a_t dt$.[1] Therefore, more effort increases the completion probability so that the current stage potentially finishes earlier.

Whether the project succeeds in each stage also depends on the agent's effort. If stage i is completed between time t and $t + dt$ and the agent exerts effort a_t during this period, then stage i has a probability $p^{(i)}(a_t)$ of succeeding and, equivalently, a probability $1 - p^{(i)}(a_t)$ of failing. We assume that

$$p^{(i)}(a) = \bar{p}^{(i)}\frac{a}{\bar{a}}, \quad a \in [0, \bar{a}],$$

where $\bar{a}$ is the maximum effort by the agent and $\bar{p}^{(i)} \in (0, 1)$ is the probability of success in stage i if the agent exerts his maximum effort.[2]

In sum, if the project is in stage i and the agent exerts effort a_t between time t and $t + dt$, this stage succeeds with probability $a_t p^{(i)}(a_t)dt$ or fails with probability $a_t \left(1 - p^{(i)}(a_t)\right) dt$ in the current period, or it continues into the next period with probability $1 - a_t dt$. We assume that project success or failure is publicly observable.

The principal recommends that the agent exert his maximum effort $\bar{a}$. However, the principal cannot continuously monitor the agent's effort, creating moral hazard in the project. If the agent shirks his effort by choosing a lower effort $a \in [0, \bar{a}]$, he enjoys a private benefit of $\lambda(\bar{a} - a)$ per unit of time. The positive constant λ measures the magnitude of moral hazard in the project. The higher the λ, the more difficult it is for the principal to incentivize the agent to exert his maximum effort. In our baseline model, we assume that the principal receives no information regarding the agent's effort. In an extension, we consider the case where the principal observes a signal about the agent's effort.

1.2 Contracting problem

During the project, the agent receives from the principal compensation whose cumulative value is described by a nondecreasing process C. The agent is protected by limited liability: The transfer dC from the principal to the agent can only be nonnegative at any time; hence, the agent cannot subsidize the project by accepting negative payment.

[1] We assume that the likelihood of completion in each stage is the same if the agent exerts the same effort. The model can be easily extended to incorporate heterogenous impacts on the completion likelihood in different stages.

[2] Our result also holds for any increasing function $p^{(i)}$ such that $ap^{(i)}(a)$ is convex in a.

When stage i is successfully completed, the principal could reward the agent a cash bonus, which we call a *milestone bonus*. When stage i succeeds at time ν, we denote the value of the milestone bonus as $R_\nu^{(i)}$. When stage i succeeds, the agent receives a new contract, which promises him a continuation utility, summarizing the value of future deferred compensation at the beginning of the next stage. This deferred compensation includes all future milestone bonuses if the project succeeds at each stage, the compensation flows dC, and the outside value if the project is terminated prematurely. When the final stage succeeds, the principal can no longer defer compensation and rewards the agent a final milestone bonus from the project payoff. The principal restricts each milestone bonus $R_\nu^{(i)}$ to be less than Δ; otherwise, the net payoff to the principal is negative even if the project succeeds. If any stage fails, the agent receives nothing at the end of that stage. Because project failure is publicly observable, we will see later that no transfer upon stage failure is without loss of generality.

The agent is protected by the limited liability against his outside value $\underline{U}$, which models the value of the agent's outside opportunity. When the agent's continuation utility reaches his outside value before the project's final completion, the project is terminated, no matter which stage the project is in, to protect the agent's limited liability.[3] This termination happens at an endogenously determined stopping time.

The agent is risk-neutral and has a subjective discounting rate r. Given a cumulative compensation C, continuation utility $U^{(i)}$ at the beginning of stage i, and milestone bonus $R^{(i)}$, the agent chooses his optimal effort to maximize his expected compensation and private benefit from shirking. The agent's value at project initiation (the beginning of the first stage) is

$$\sup_{a\in[0,\bar{a}]} \mathrm{E}^a\left[\int_0^{\tau\wedge\nu^{(1)}} e^{-rs}\left(dC_s + \lambda(\bar{a} - a_s)ds\right) + \mathbb{1}_{\{\nu^{(1)}\le\tau\}} e^{-r\nu^{(1)}} U^c_{\nu^{(1)}} + \mathbb{1}_{\{\nu^{(1)}>\tau\}} e^{-r\tau}\underline{U}\right], \tag{1}$$

where $\nu^{(1)}$ is the stage 1 completion time, and τ is the project's endogenous termination time.[4] Before $\tau\wedge\nu^{(1)} = \min\{\tau,\nu^{(1)}\}$, the agent receives compensation dC and the shirking benefit. If stage 1 is completed before project termination, i.e., $\nu^{(1)}\le\tau$, the agent receives the continuation utility $U^{(2)}_{\nu^{(1)}}$ to start stage 2 and a milestone bonus $R^{(1)}_{\nu^{(1)}}$ upon stage 1's success, which happens with probability $p^{(1)}(a_\nu)$. Therefore,

$$U^c_{\nu^{(1)}} = p^{(1)}(a_\nu)\left(U^{(2)}_{\nu^{(1)}} + R^{(1)}_{\nu^{(1)}}\right)$$

[3] We will show in the proof of Lemma 2 that this is also the agent's optimal time to quit the project if he has the discretion to choose his time to quit any time before his limited liability constraint is binding.

[4] $\mathbb{1}_A$ is an indicator function that equals 1 when A happens and 0 otherwise.

is the expected continuation value upon stage 1 completion.[5] When $n^{(1)} > t$, the project is terminated before its completion and the agent receives his outside value $\underline{U}$ at t.

We assume that the principal has the commitment power to issue a long-term contract for the agent. A contract is a triplet $(C, U^{(i)}, R^{(i)}; 1 \le i \le N)$, where C is nondecreasing and $U^{(i)}$ and $R^{(i)}$ are nonnegative processes. The principal considers the class of contracts that will incentivize the agent to exert his maximum effort $\bar{a}$. The principal is assumed to be risk-neutral with a subjective discounting rate r.

The principal's contracting problem is

$$\sup_{U^{(i)},R^{(i)},C} \mathbb{E}^{\bar{a}}\Big[\int_0^{t\wedge n^{(1)}} e^{-rs}\, dC_s + 1_{\{n^{(1)}\le t\}} e^{-rn^{(1)}} V^c_{n^{(1)}} + 1_{\{t<n^{(1)}\}} e^{-rt} L\Big], \tag{2}$$

subject to (i) the incentive compatible constraint that $\bar{a}$ is the agent's optimal effort for the problem (1) and (ii) the agent's participation constraint that the agent's value at time 0 is at least $\underline{U}$. In (2), if stage 1 is completed before project termination, the principal's expected value at stage 1 completion is

$$V^c_{n^{(1)}} = p^{(1)}\left(V^{(2)}_{n^{(1)}} - R^{(1)}_{n^{(1)}}\right), \tag{3}$$

where $V^{(2)}_{n^{(1)}}$ is the principal's value at the beginning of stage 2. Therefore, $V^c_{n^{(1)}}$ is the expected continuation value net of the milestone bonus $R^{(1)}_{n^{(1)}}$.[6] If the project is terminated before stage 1 completion, the principal only receives a scrap value L from the project and does not pay the agent at the termination.

The principal's value in (2) is recursively defined, with the stage i value defined similarly to (2) and the final stage value as follows:

$$V_t^{(N)} = \sup_{R^{(N)},C} \mathbb{E}_t^{\bar{a}}\Big[\int_t^{n^{(N)}} e^{-r(s-t)}\, dC_s + 1_{\{n^{(N)}\le t\}} e^{-r(n^{(N)}-t)} p^{(N)}\left(D - R^{(N)}_{n^{(N)}}\right) + 1_{\{t<n^{(N)}\}} e^{-r(t-t)} L\Big]. \tag{4}$$

In (4), $t \ge n^{(N-1)}$ and $\mathbb{E}_t^{\bar{a}}[\cdot] = \mathbb{E}^{\bar{a}}[\cdot \mid \mathcal{F}_t^P]$ is the conditional expectation with respect to the

[5] To simplify notation, we assume that the agent receives nothing upon project failure. If the agent receives his outside value $\underline{U}$, the principal would adjust the continuation utility $U^{(2)}$ and milestone bonus $R^{(1)}$ accordingly to achieve the same expected continuation value at the completion of the stage 1.

[6] In addition to milestone bonuses at stage success, the principal can also pay the agent a severance package at failure. The principal needs to restrict the severance pay to be less than the success reward in order to incentivize the highest probability of success; see the proof of Lemma 2. However, both the agent's and the principal's values depend on the expected reward (from both deferred compensation and milestone bonuses) at stage completion. With severance pay at failure, if it achieves the same expected reward for the agent, it induces the same value to the principal. Therefore, it is without loss of generality to assume that the principal only rewards project success and does not pay upon failure.

principal's information filtration $\{F_t^P\}$, which is generated by the sequence of completion time $\{n^{(i)}; i \geq 1\}$. In (4), when the final stage succeeds at time $n^{(N)}$, the principal receives the project payoff D net of the final milestone bonus $R^{(N)}_{n^{(N)}}$.

We assume that

$$r < r,$$

indicating that the principal is more patient than the agent. This technical condition ensures the existence of an optimal contract so that the principal does not defer the compensation forever. The same condition is required by DeMarzo and Sannikov (2006).

2 Optimal contract in a baseline model

2.1 The agent's problem

Given a contract $(C, U^{(i)}, R^{(i)}; 1 \leq i \leq N)$, we first solve the agent's problem (1) in this section. When the agent exerts high effort, he loses the shirking benefit. When the current stage is completed, the agent loses his current continuation utility, which would continue if he were to prolong the project, but he would receive a reward from the next contract as well as a milestone bonus if the current stage succeeds. Only when the expected reward is large enough will the agent exert his maximum effort.

Suppose that the project is in stage i. Conditioning on no completion before t, i.e., $t \in (n^{(i-1)}, n^{(i)}]$, the agent's continuation utility at time t is defined as

$$U_t = \sup_{a \in [0,\bar{a}]} \mathrm{E}_t^a \Big[\int_t^{t \wedge n^{(i)}} e^{-r(s-t)} \big(dC_s + l(\bar{a} - a_s)ds \big) + \mathrm{I}_{\{n^{(i)} \leq t\}} e^{-r(n^{(i)}-t)} U^c_{n^{(i)}} + \mathrm{I}_{\{t < n^{(i)}\}} e^{-r(t-t)} \underline{U} \Big], \tag{5}$$

where $U^c_{n^{(i)}} = p^{(i)}(a_{n^{(i)}}) \big(U^{(i+1)}_{n^{(i)}} + R^{(i)}_{n^{(i)}} \big)$ is the expected continuation value at stage completion time $n^{(i)}$. In (5), the conditional expectation is with respect to the agent's filtration, which is generated by $\{n^{(i)}, a; i \geq 1\}$ as well as the event $\{n^{(i)} \leq t\}$.

The following result characterizes the dynamics of the agent's continuation utility and the optimal effort.

Lemma 1. *The agent's continuation utility U follows the dynamics*

$$dU_t = rU_t dt + \inf_{a_t \in [0,\bar{a}]} \Big\{ a_t U_t - l(\bar{a} - a_t) - a_t U^c_t \Big\} dt - dC_t, \tag{6}$$

where $U^c_t = p^{(i)} \big(U^{(i+1)}_t + R^{(i)}_t \big)$ for $t \in [n^{(i-1)}, n^{(i)})$, and $U^{(N+1)} \equiv 0$. The agent's optimal effort *is given by*

$$a_t = \begin{cases} \bar{a}, & p^{(i)} \quad U_t^{(i+1)} + R_t^{(i)} \qquad l + U_t, \\ 0, & \textit{otherwise.} \end{cases} \tag{7}$$

When $U_t^c \quad l + U_t$, *U follows*

$$dU_t = \quad (r + \bar{a})U_t \quad \bar{a}_t U_t^c \;\; dt \quad dC_t. \tag{8}$$

Lemma 1 shows that the agent's optimal effort is determined by three forces:

(i) If the agent exerts effort a, he earns private benefit $l(\bar{a} \quad a_t)dt$ from shirking.

(ii) The agent's effort also changes the likelihood of stage completion. If the agent exerts effort a, the current stage completes with probability adt in the next infinitesimal period of time. Therefore, the agent's expected reward from completion is $adt\, U_t^c$.

(iii) Once the current stage is completed, the agent also forfeits the current continuation utility U_t; hence, the agent's expected loss from stage completion is $adtU_t$.

To incentivize the agent's maximum effort, the principal needs to ensure that the marginal benefit of effort U_t^c is larger than the marginal cost $l + U_t$.

2.2 Baseline model

In this section, we consider a baseline model in which the principal only observes the completion time of the various stages of the project and uses deferred compensation and milestone bonuses to incentivize effort. After building intuition from a single-stage project, we examine the optimal contract in a two-stage project.

2.2.1 Single-stage project

We consider a single-stage project in this section. To simplify notation, we suppress the superscript (1) throughout this section. We assume that project termination is inefficient,

$$(r + \bar{a})L + (r + \bar{a})\underline{U} < \bar{a}p\mathrm{D}. \tag{9}$$

This condition is equivalent to $L < \frac{1}{r+\bar{a}} \;\; \bar{a}p\mathrm{D} \quad (r + \bar{a})\underline{U}$. This means that the project's liquidation value L is less than the net present value of the project's expected payoff, net paying the agent a constant cash flow of $(r + \bar{a})\underline{U}$, which has a present value of $\underline{U}$ under the agent's effective discount rate $r + \bar{a}$. We also assume

$$\underline{U} < \frac{\bar{a} I}{r} \tag{10}$$

so that the agent's outside option is not better than shirking on the project.

Lemma 1 implies that the expected reward must satisfy the following to incentivize the agent to exert his maximum effort:

$$pR_t \geq I + U_t. \tag{11}$$

Thus, the marginal expected reward from completion is weakly larger than the marginal gain of shirking and delay, in order to incentivize the agent to exert his maximum effort. The principal's value function in (4), conditioning on no project completion before t, i.e., $n \geq t$, is

$$V_t = \sup_{R,C} \mathrm{E}_t^{\bar{a}} \Big[\int_t^{t \wedge n} e^{-r(s-t)} dC_s + \mathbb{1}_{\{n \leq t\}} e^{-r(n-t)} p \left(D - R_n\right) + \mathbb{1}_{\{t < n\}} e^{-r(t-t)} L \Big], \tag{12}$$

where R is subject to the constraint (11).

We now derive the differential equation that V satisfies. Consider the compensation $dC_t = c_t dt$ for some nonnegative and potentially unbounded c_t. It follows from the dynamic programming that V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(r + \bar{a})V = \sup_{R,c} \Big\{ \bar{a}p(D - R) + \big((r + \bar{a})U - \bar{a}pR\big) V' - (1 + V')c \Big\}, \tag{13}$$

where $pR \geq I + U$ and $R \leq D$. When U reaches the agent's outside value $\underline{U}$, the project is terminated and the principal receives L. Therefore, V satisfies the boundary condition

$$V(\underline{U}) = L. \tag{14}$$

On the left-hand side of (13), $r + \bar{a}$ is the principal's effective discounting rate. On the right-hand side, $\bar{a}p(D - R)$ is the expected net payoff in dt time, and $\big((r + \bar{a})U - \bar{a}pR\big) V'$ is the marginal change in the principal's value due to changes in agent's continuation utility before compensation. The term $-(1 + V')c$ measures the net value of comepsnation and it is linear in c. Because c can be unbounded, in order for the right-hand side of (13) to have a finite value, it is necessary for $1 + V' \geq 0$. Consider $-V'$ to be the marginal benefit of paying the agent. Whenever $1 > -V'$, the marginal cost of paying the agent is larger than the marginal benefit, and the principal defers the payment by choosing $c = 0$ until $V' = -1$. Define

$$\overline{U} = \inf\{U \geq \underline{U} : V' \leq -1\}. \tag{15}$$

To understand the principal's optimal choice for the success reward, we start with the first best case, where the agent's action is contractible and $L = L^{fb} := \frac{1}{r+\bar a}\left(\bar a p D - (r+\bar a)\underline{U}\right)$. In that case, a linear value function V with $V(\underline{U}) = L$ and $V' \equiv -1$ corresponds to the first best value. If the principal rewards the agent with bonus R upon project success, the principal's expected payoff decreases by $\bar a pRdt$. Meanwhile, (8) with $U^c = pR$ therein implies that the agent's continuation utility decreases by $\bar a pRdt$, which impacts the principal's utility by $-\bar a pRV' dt$. Because $V' \equiv -1$, the principal's net value change is zero for any R. Therefore, the choice of R is irrelevant to the principal in the first best case. The agent is compensated only by C in this case.

When $L < L^{fb}$, termination is inefficient and the principal's choice of R matters. The optimization of R in (13) is

$$\sup_R \bar a \left\{ -pR - pRV' \right\},$$

subject to $pR \geq I + U$. When the project is completed, pR is the expected cost and $-pRV'$ is the expected benefit of paying the agent milestone bonus R. Because $1 > -V'$ when $U < \overline{U}$, the expected cost always dominates the expected benefit. Therefore, the principal rewards the agent the minimum amount to satisfy the incentive compatibility constraint, i.e., the optimal milestone bonus satisfies

$$pR^* = I + U. \tag{16}$$

The following proposition summarizes the optimal contract and the agent's continuation utility dynamics under the optimal contract.

Proposition 1. *Suppose that (9) and (10) hold. The principal chooses the initial value of the agent's continuation utility U_0 to maximize her value function V. The agent's continuation utility decreases over time until it reaches the outside value $\underline{U}$, at which time the contract is terminated if the project is still not completed. If the project is successfully completed before U reaches $\underline{U}$, the agent is rewarded with milestone bonus $R^* = \frac{1}{p}(I + U)$. The principal pays no cash compensation C in the process.*

The optimal contract features a milestone bonus, no cash compensation, and a hard deadline. As the agent's continuation utility decreases over time, the size of the milestone bonus also decreases with time. Because the marginal benefit of cash compensation is always less than the marginal cost, the principal never pays the agent cash compensation. Combining (8), (16), and $dC_t \equiv 0$, we find that the termination time t^* of the project is a constant:

$$t^* = \frac{1}{r}\log\frac{\bar a I - rU_0}{\bar a I - r\underline{U}}. \tag{17}$$

This is a hard deadline for the project. It is decreasing in $\bar a$, I, $\underline{U}$ and increasing in U_0. When $\bar a$

increases, the expected completion time $1/\bar{a}$ decreases; when l is larger, the agency friction is more severe; when U_0 decreases or $\underline{U}$ increases, the agent has less skin in the project. In all cases, the principal shortens the hard deadline.

To illustrate the quantitative implications of the model, we solve the HJB equation (13) with its boundary condition (14) numerically and present several examples of the principal's value in Figure 1.

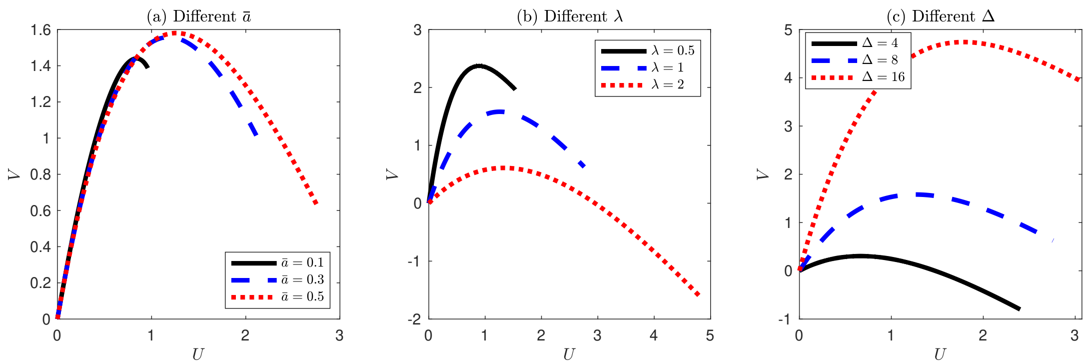


Figure 1. **Optimal contract in a single-stage project**

Notes: This figure plots the principal's value for different maximal effort, private benefits, and project payoffs. Each curve ends at $\overline{U}$*. If not listed explicitly in the legend, parameters used are* $r = 0.1$*,* $r = 0.05$*,* $\bar{a} = 0.5$*,* $l = 1$*,* $D = 8$*,* $p = 0.5$*,* $\underline{U} = 0$*, and* $L = 0$*.*

At the project inception, the agent's continuation utility starts from $U_0 = \text{argmax} V(U)$. In our contract implementation, $U_0 \quad \underline{U}$ measures the maximum balance of a credit line that the principal grants the agent. When the credit line is maxxed out, the project is terminated. Figure 1, Panel (a) shows that the larger the agent's maximum effort $\bar{a}$ is, the longer the credit line is. When $\bar{a}$ is larger, the project is more likely to be completed; consequently, the principal grants the agent more financial flexibility to allow the agent to be successful. When the agent's private benefit of shirking is larger, the principal needs to offer a bigger milestone bonus to incentivize the agent. Hence, the principal's value function decreases with l, as Panel (b) shows. The principal also offers a longer credit line for larger l. When the project payoff D is larger, Panel (c) shows that the principal offers a longer credit line to increase the chances of project success before termination.

2.2.2 Multiple-stage project

In this section, we study a two-stage project ($N = 2$). The first stage has to be completed successfully before the second stage begins. The principal only receives the project payoff when the second stage succeeds. However, the principal can terminate the project if the first stage fails. When the first stage is successfully completed, the principal determines how much deferred compensation to promise the agent in the second stage, in the form of a second-stage milestone bonus and cash compensation C, and whether to reward the agent a first-stage milestone bonus for successfully completing the first stage.

After successful completion of the first stage, the second stage contracting problem is the same as the one described in Section 2.2.1. We focus on the first-stage problem in what follows. When the first stage is completed at time $n^{(1)}$, the agent receives a new contract that promises him, at the beginning of the second stage, a continuation utility $U^{(2)}_{n^{(1)}}$, summarizing the value of deferred compensation in the second stage. Moreover, the principal could also reward the agent a milestone bonus $R^{(1)}_{n^{(1)}}$.

Lemma 1 shows that the agent exerts effort $\bar{a}$ if and only if

$$p^{(1)} \left(U_t^{(2)} + R_t^{(1)}\right) \geq I + U_t. \tag{18}$$

Equation (2) shows that the principal's value in the first stage is

$$V_t^{(1)} = \sup_{U^{(2)},R^{(1)},C} \mathrm{E}_t^{\bar{a}}\left[\int_t^{n^{(1)}\wedge t} e^{-r(s-t)}(-dC_s) + 1_{\{n^{(1)}\leq t\}} e^{-r(n^{(1)}-t)} V^c_{n^{(1)}} + 1_{\{t<n^{(1)}\}} e^{-r(t-t)} L\right], \tag{19}$$

subject to (18), $R^{(1)} \in [0, \mathrm{D}]$, and $U^{(2)} \in [\underline{U}, \overline{U}^{(2)}]$. In (19), $V^c_{n^1}$ is given in (3), the milestone bonus $R^{(1)}$ is restricted to be at most the project payoff D, and the second-stage initial utility $U^{(2)}$ is at most the second-stage maximum utility $\overline{U}^{(2)}$.

An argument similar to that in Section 2.2.1 yields the following HJB equation for $V^{(1)}$:

$$(r+\bar{a})V^{(1)} = \max_{U^{(2)},R^{(1)}} \left\{ \bar{a}p^{(1)} \left(V^{(2)}(U^{(2)}) - R^{(1)}\right) + \left((r+\bar{a})U - \bar{a}p^{(1)}\left(U^{(2)}+R^{(1)}\right)\right)(V^{(1)})' \right\}, \tag{20}$$

subject to (18), $R^{(1)} \in [0, \mathrm{D}]$, and $U^{(2)} \in [\underline{U}, \overline{U}^{(2)}]$, and the boundary conditions

$$V^{(1)}(\underline{U}) = L \quad \text{and} \quad (V^{(1)})'(\overline{U}^{(1)}) = -1, \tag{21}$$

for some $\overline{U}^{(1)}$.

We first discuss the choice of $U^{(2)}$ and $R^{(1)}$ at the successful completion of the first stage.

The first-order condition of $U^{(2)}$ in (20) is

$$\bar{a}p^{(1)} \quad (V^{(2)})'(U^{(2)}) \quad (V^{(1)})'(U) \quad = 0, \tag{22}$$

and the first-order condition of $R^{(1)}$ is

$$\bar{a}p^{(1)} \quad 1 + (V^{(1)})'(U) \quad = 0. \tag{23}$$

Because $(V^{(2)})' > \quad 1$ when $U^c < \overline{U}^{(2)}$, (22) and (23) cannot be satisfied simultaneously: When $U^{(2)}$ is an interior, the optimal $R^{(1)}$ must be zero; $R^{(1)}$ is positive only when $U^{(2)}$ reaches its maximum value $\overline{U}^{(2)}$.

This result creates a pecking order for the principal to reward the agent for first-stage success. The principal prefers to increase the agent's deferred compensation in the second stage over paying the agent a milestone bonus. This preference is because the principal's marginal benefit of cash bonus $\quad (V^{(2)})'(U^{(2)})$ is less than the unit marginal cost when $U^{(2)} < \overline{U}^{(2)}$. Only when $U^{(2)}$ reaches $\overline{U}^{(2)}$ does the principal combine the second-stage deferred compensation with a milestone bonus at the end of the first stage.

The following proposition summarizes the optimal deferral compensation and milestone bonus for a two-stage project.

Proposition 2. *Suppose that* (9) *and* (10) *hold. Then the following statements hold:*

(i) The principal pays milestone bonus $R^{(1)} = \frac{1}{p^{(1)}}(I + U) \quad \overline{U}^{(2)}$ *only when* $U \quad p^{(1)}\overline{U}^{(2)} \quad I$.

(ii) When the first stage is successfully completed, the agent's continuation utility jumps up to $U^{(2)}$ *with* $U^{(2)} \quad \overline{U}^{(2)}$*, and he receives milestone bonus* $R^{(1)}$ *when* $U^{(2)} = \overline{U}^{(2)}$.

(iii) $V^{(1)}(U) < V^{(2)}(U)$ *for* $\underline{U} < U \quad \min \quad \overline{U}^{(1)}, \overline{U}^{(2)}$. *Therefore, the agent's equity stake* $\frac{U}{U+V^{(1)}(U)}$ *in the first stage is always larger than his equity stake* $\frac{U}{U+V^{(2)}(U)}$ *in the second stage.*

This result shows that the milestone bonus strategy is a threshold type. Only when U is sufficiently large and $U^{(2)}$ reaches its maximum value does the principal reward the agent with a milestone bonus. Given the agent's continuation utility U and the principal's value $V(U)$, we define the agent's equity stake as the ratio between the agent's continuation utility from deferred compensation and the project's enterprise value $U + V(U)$. Proposition 2 (iii) shows that the agent's equity stake is always larger in the first stage. The intuition behind this result is the following: The project may fail at the end of the first stage, in which case the project payoff is zero. Moreover, the second-stage enterprise value is the discounted future project payoff net compensation such that the expected payoff from the first stage is less than the expected payoff from the second stage, i.e.,

$$p^{(1)}\left(V^{(2)}(U^{(2)}) + U^{(2)}\right) < p^{(2)}D.$$

With less expected payoff, the principal's value in the first stage $V^{(1)}(U)$ is less than her value in the second stage $V^{(2)}(U)$, when the agent has the same amount of skin U in the project. Therefore, the principal shares a larger proportion of the project's enterprise value in the second stage, resulting in a smaller equity stake for the agent.

The result that the agent's equity stake is smaller in later stages is consistent with evidence from empirical studies. Bengtsson and Sensoy (2015) observe that entrepreneurs' equity stakes in startups are lower in later funding rounds. Ewens, Nanda, and Stanton (2020) document that an entrepreneur's equity stake decreases with the firm's age. An explanation for this phenomenon is that additional funding injections in later financing rounds dilute the entrepreneur's equity stake. Our result shows that even without additional funding injections, resolution of uncertainty at the end of each stage increases the project value, thereby decreasing the entrepreneur's equity stake.

Figure 2 shows that the principal's value is larger in the second stage (Panel (a)) and the agent's equity stake is higher in the first stage (Panel (b)). At the beginning of the first stage, $(U_0, V^{(1)}(U_0))$ starts from the highest point of the red dotted line in Panel (a), then moves to the left along the red dotted line. If the first stage is successful before U becomes lower than 0.5, which is the horizontal coordinate of the kink point in Panel (c), $(U, V^{(1)}(U))$ jumps to the right end of the black solid line $(\overline{U}^{(2)}, V^{(2)}(\overline{U}^{(2)}))$ in Panel (a) to start the second stage, and the agent receives a milestone bonus whose value is shown by the red dotted line in Panel (d). If the first stage is successful when $U < 0.5$, $(U, V^{(1)}(U))$ jumps to $(U^{(2)}, V^{(2)}(U^{(2)}))$, with $U^2 = U^2(U)$ given in Panel (c), on the black solid line in Panel (a) and the second stage starts from there, but the agent receives no milestone bonus. If the first stage fails, the project is terminated. In the second stage, $(U, V^{(2)}(U))$ moves to the left along the black solid line in Panel (a). In both stages, if U reaches zero before stage completion, the project is terminated. The principal grants a credit line with limit U_0 in the first stage and extends the credit line to the limit $U^{(2)}$ in the second stage. The principal gives a much longer credit line in the second stage to allow ample time for the agent to complete the final stage. When the final stage succeeds, the agent receives the final milestone bonus $R^{(2)}(U)$, whose value is indicated by the solid black line in Panel (d).

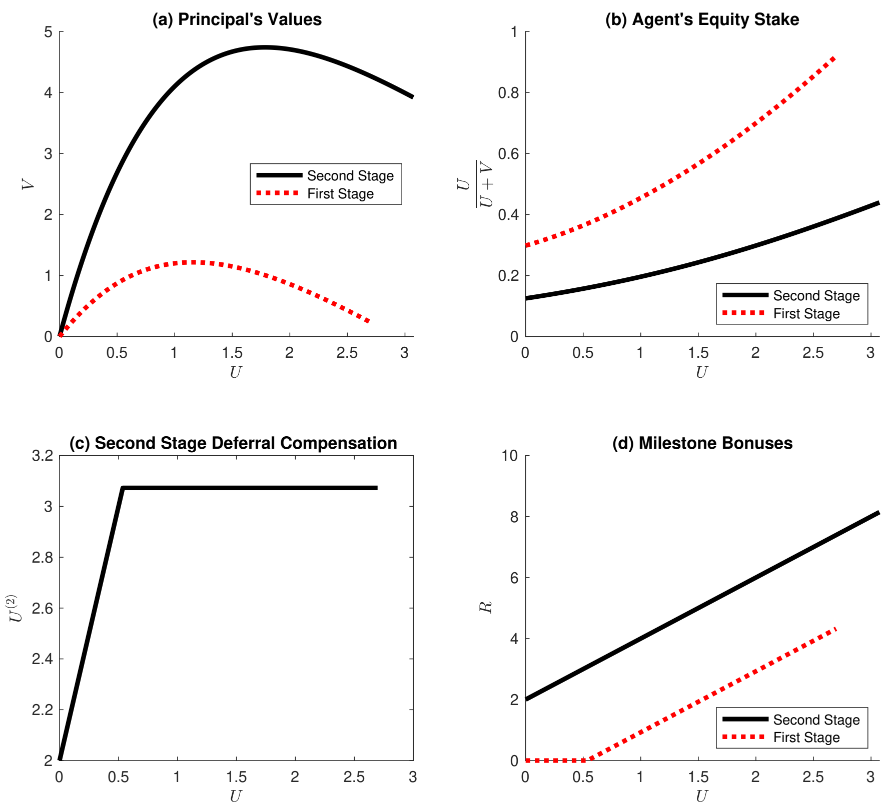


Figure 2. **Optimal contract in a two-stage project**

Notes: This figure plots the principal's values, the agent's equity stake, and milestone bonuses in both stages as well as the agent's continuation utility $U^{(2)}$ *at the beginning of the second stage. Parameters used are* $r = 0.1$, $r = 0.05$, $\bar{a} = 0.5$, $I = 1$, $D = 16$, $p^{(1)} = p^{(2)} = 0.5$, $\underline{U} = 0$, *and* $L = 0$.

3 Monitoring and reward

In the baseline model, we assume that the principal does not receive any information on the agent's effort. In this section, we consider a natural extension in which the principal observes a noisy signal about the agent's effort. This noisy signal can be thought of as the agent's working hours or regularly submitted progress reports. With this additional signal, the principal can both contract on the signal and reward the agent with milestone bonuses or deferred compensation. In the optimal contract, contracting on the signal and reward complement each other. When positive (negative) evidence about the agent's effort accumulates, the optimal contract increases (decreases) the magnitude of the reward and decreases (increases) the contract sensitivity on the signal. After examining a single-stage project in this extension, we explore the optimal contract for a two-stage project. We show that the main insight from the baseline model still holds in this extension.

We assume that the principal observes a noisy signal y about the agent's effort. The dynamics of y follow

$$dy_t = a_t dt + s dB_t^a, \tag{24}$$

where B^a is a standard Brownian motion under the probability measure P^a induced by the agent's effort. We assume that B^a is independent of the Poisson process driving the completion time of the different stages of the project as well as the random variables indicating the success or failure of each stage. The principal can contract on the signal y and can also determine $(U^{(i)}, R^{(i)}; 1 \le i \le N)$ at the completion of each stage as in the previous section.

The following result extends Lemma 1 to the current setting.

Lemma 2. *The agent's continuation utility U follows the dynamics*

$$dU_t = rU_t dt + \inf_{a_t \in [0,\bar{a}]} \left\{ a_t U_t - l(\bar{a} - a_t) - a_t U_t^c - j_t a_t \right\} dt + j_t dy_t - dC_t, \tag{25}$$

for some process j. In (25), $U_t^c = p^{(i)} \left(U_t^{(i+1)} + R_t^{(i)} \right)$, when $t \in [n^{(i-1)}, n^{(i)})$, and $U^{(N+1)} \equiv 0$. The agent's optimal effort is given by

$$a_t = \begin{cases} \bar{a}, & U_t^c + j_t \ge l + U_t, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

When $U_t^c + j_t \ge l + U_t$, U follows

$$dU_t = \left((r + \bar{a})U_t - \bar{a} U_t^c \right) dt + j_t s_t dB_t^{\bar{a}} - dC_t. \tag{27}$$

Lemma 2 shows that the agent's continuation utility varies with signal y. The principal chooses the contract sensitivity j. To incentivize maximum effort, the principal needs to choose a sufficiently large j so that the sum of j and the marginal expected reward U^c for completing the current stage dominates the marginal cost $l + U$. The contract sensitivity j and the expected reward U^c complement each other to incentivize the agent's maximum effort. The optimal choice of j and the continuation utility, together with milestone bonuses $(U^{(i)}, R^{(i)}; 1 \le i \le N)$, will be determined in the optimal contract.

3.1 Single-stage project

Consider the principal's problem (12) in a single-stage project.[7] Following the strategy in the previous section, we first derive the HJB equation that is satisfied by V. Consider $dC_t = c_t dt$, where c_t is nonnegative and could be unbounded. It follows from (27) with $U^c = pR$ therein and the dynamic programming that V satisfies the following HJB equation:

$$(r+\bar{a})V = \sup_{R,c,j}\left\{\bar{a}p(D-R) + \left((r+\bar{a})U - \bar{a}pR\right)V' - (1+V')c + \tfrac{1}{2}j^2 s^2 V''\right\}, \tag{28}$$

subject to $R \in [0, D]$, $c \geq 0$, and $j \geq l + U_t - pR$.

An argument similar to that in Section 2.2.1 shows that $V' > -1$ until a threshold $\overline{U}$ in (15). Due to the presence of Brownian noise, the optimal compensation dC is a reflecting type. Once U reaches $\overline{U}$, the principal makes a minimum payment to reflect U being lower than $\overline{U}$. The boundary $\overline{U}$ is pinned down by the boundary conditions

$$V'(\overline{U}) = -1 \quad \text{and} \quad V''(\overline{U}) = 0, \tag{29}$$

where the second boundary condition is a super-contact condition. When $U = \underline{U}$, V satisfies the boundary condition

$$V(\underline{U}) = L. \tag{30}$$

The choice of C discussed above reduces the HJB equation (28) to

$$(r+\bar{a})V = \sup_{R,j}\left\{\bar{a}p(D-R) + \left((r+\bar{a})U - \bar{a}pR\right)V' + \tfrac{1}{2}j^2 s^2 V''\right\}, \tag{31}$$

subject to $R \in [0, D]$, $j \geq l + U - pR$ and boundary conditions (29) and (30).

The following proposition summarizes the principal's optimal choice in regard to milestone bonuses, contract sensitivity, and compensation.

Proposition 3. *Suppose that* (9) *and* (10) *hold.*

(i) The optimal reward is

$$R^* = \min\left\{\max\left\{\frac{1}{p}\left(l + U + \frac{\bar{a}(1+V')}{s^2 V''}\right), 0\right\}, D\right\}. \tag{32}$$

The optimal contract sensitivity is $j^* = l + U - pR^*$, *and it is always positive.*

(ii) When U *approaches* $\overline{U}$, R^* *converges to* $\min\{(l+\overline{U})/p, D\}$. *When* R^* *is an interior*

[7] We omit the superscript (1) throughout this section to simplify notation.

optimal level, the optimal contract sensitivity φ^* *converges to zero.*

(iii) *The principal's optimal value function* V *is concave.*

(iv) *The optimal compensation* C *reflects* U *to keep it lower than* $\overline{U}$ *once* U *reaches* $\overline{U}$.

Recall from assumption (9) that project termination is inefficient. Similar to the baseline case, the principal's choice of R matters. The optimal choice R^* is presented in Proposition 3 (i). To understand the intuition, we start from the concavity of V in (iii). The inefficient termination of the project before its completion introduces endogenous risk aversion to the risk-neutral principal; hence, the principal is averse to future variations of U. Due to $V'' < 0$, the principal chooses the minimal contract sensitivity $\varphi^* = \lambda + U - \pi R^*$ to incentivize maximum effort $\bar{a}$ from the agent. Therefore, when we plug $\varphi^* = \lambda + U - \pi R$ into the right-hand side of (31), the terms containing R are

$$\max_{R} \left\{ -\bar{a}\pi R(1 + V') + \tfrac{1}{2}(\lambda + U - \pi R)^2 \sigma^2 V'' \right\}. \tag{33}$$

In this optimization problem, $\bar{a}\pi R$ is the expected direct cost and $-\bar{a}\pi R V'$ is the expected benefit from paying the reward to the agent; hence, $-\bar{a}\pi R(1 + V')$ is the net marginal benefit of the reward. This term is the same as the baseline model. The additional term $\frac{1}{2}(\lambda + U - \pi R)^2 \sigma^2 V''$ is the risk aversion cost to the principal arising from exposing the agent's continuation utility to the noisy signal. Therefore, the interior optimal reward is determined by the mean-variance trade-off:

$$\lambda + U - \pi R^* = -\frac{\bar{a}(1 + V')}{\sigma^2 V''}.$$

When U is close to the agent's outside value $\underline{U}$, the project is close to an inefficient termination. As our numeric result later shows, R^* is low in this case. To illustrate the intuition, recall the dynamics of the agent's continuation utility U from (27): The drift of U is $(r + \bar{a})U - \bar{a}\pi R^*$, and the volatility is $(\lambda + U - \pi R^*)\sigma$. When U is close to $\underline{U}$, a large bonus R^* would induce a negative drift to the agent's continuation utility, which leads U closer to $\underline{U}$ and the project nearer to its inefficient termination. To mitigate this termination risk, the principal chooses a small value of R^* when U is close to the termination boundary $\underline{U}$ so that a positive drift leads the continuation utility U away from $\overline{U}$.

When U is close to the payment boundary $\overline{U}$, both terms in (33) converge to zero due to the boundary condition (29). However, we show in the proof of Proposition 3 that the variance term dominates in the sense that

$$\lim_{U \to \overline{U}} \frac{1 + V'}{V''} = 0.$$

As a result, the principal chooses $j = I + \overline{U} - pR = 0$ to minimize the variance term. This is achieved by choosing R converging to $(I + \overline{U})/p$ if R has not reached its upper bound D. In this case, U is sufficiently far away from the termination boundary $\underline{U}$, and the threat of inefficient termination becomes a secondary consideration for the principal.

We saw in Lemma 2 that the contract sensitivity j and R complement each other to incentivize the agent's maximum effort. As Proposition 3 shows, the bonus R plays an increasingly important role as the agent's continuation utility increases. When U is close to $\underline{U}$, the principal exposes the compensation more to the signal and less to the milestone bonus in order to incentivize the agent's maximum effort. As the agent's continuation utility increases, the optimal contract increasingly depends on the milestone bonus. At the limit, as $U \to \overline{U}$, the agent's effort is only incentivized by the milestone bonus, and the contract is insensitive to the signal.

Due to the positive contract sensitivity, the agent's continuation utility is positively correlated with Brownian shocks. As a result, when the signal indicates that the agent is working hard, the principal tunes down the contract sensitivity so that the agent's continuation utility is less sensitive to the signal, but the principal increases the upside reward to motivate the agent. However, when negative evidence accumulates against the agent, the principal reduces the upside reward and switches to more intensive monitoring by increasing the contract's sensitivity to the signal. As a result, further negative shocks to the signal reduces the agent's continuation utility by an increasing magnitude and the agent is punished more harshly.

To understand the quantitative implications of the model, we solve the HJB equation (31) numerically, together with boundary conditions (29) and (30) and present the optimal contract in Figure 3.

Panels (a) and (b) in Figure 3 show that the optimal contract sensitivity j decreases to zero and the optimal success reward R increases to $(I + \overline{U})/p$ when U increases to $\overline{U}$. These results indicate that the principal dynamically switches from monitoring to reward as the agent's continuation utility increases.

Figure 4 compares the optimal contract sensitivities and milestone bonuses under different parameter configurations. In the contract implementation, $\overline{U} - \underline{U}$ measures the maximum limit of a credit line. When the limit of the credit line is reached, the project is terminated. Therefore, $\overline{U} - \underline{U}$ represents the financial flexibility that the principal grants the

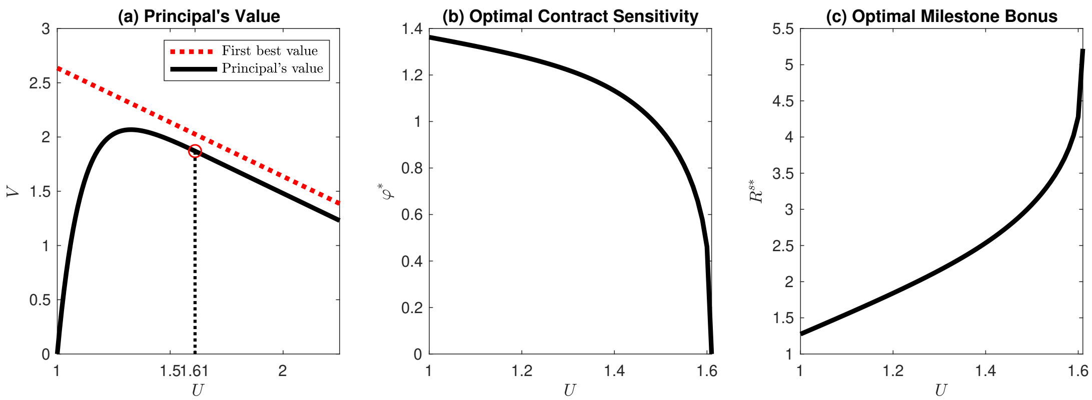
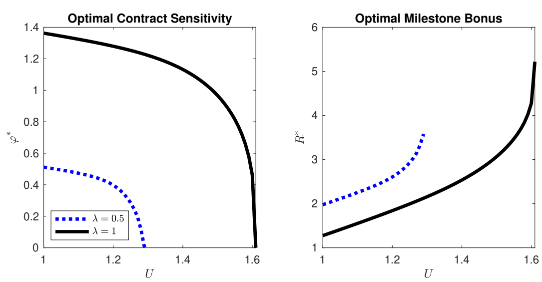
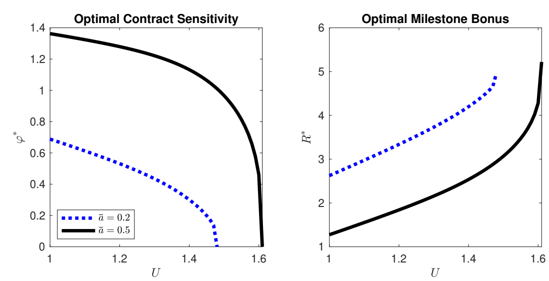


Figure 3. **Optimal contract sensitivity and milestone bonus in a single-stage project**

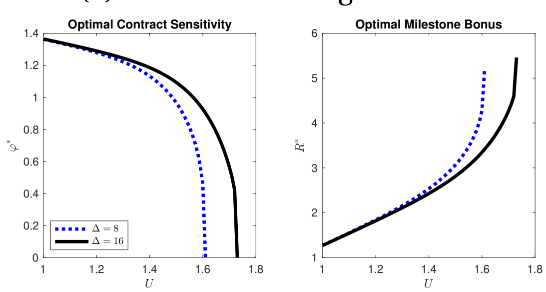
Notes: This figure plots the principal's value, the optimal contract sensitivity, and the optimal milestone in a single-stage project. Parameters used are $r = 0.1$, $r = 0.05$, $s = 0.2$, $\bar{a} = 0.5$, $l = 1$, $D = 8$, $p = 0.5$, $\underline{U} = 1$, *and* $L = 0$. *The agent's outside value* $\underline{U}$ *is chosen to be* 1 *to produce an interior optimal* R . *Panel (a) shows that* $\overline{U} = 1.61$.



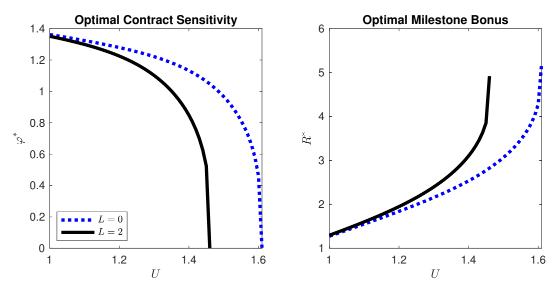
(a) Different shirking benefit l



(b) Different maximum effort $\bar{a}$



(c) Different project payoff D



(d) Different liquidation value L

Figure 4. **Optimal contract sensitivity and milestone bonus for different l, $\bar{a}$, D, and L**

Notes: If not listed in the legend, parameters used are $r = 0.1$, $r = 0.05$, $s = 0.2$, $\bar{a} = 0.5$, $l = 1$, $D = 8$, $p = 0.5$, $\underline{U} = 1$, *and* $L = 0$.

agent. Panel (a) in Figure 4 shows that the contract sensitivity has a larger magnitude when the agent's private benefit from shirking is larger. When negative evidence is accumulating against the agent and the project is close to its termination, the milestone bonus is also smaller for the agent with a higher benefit from shirking.

Panel (b) of Figure 4 shows that the principal monitors the agent with higher maximum effort more intensely. If we interpret the agent's maximum effort as his ability, then the principal monitors a more capable agent more closely and chooses a higher contract sensitivity. This result may sound counter-intuitive, but the reason behind this result is as follows: For a fixed milestone bonus R, the expected cost $\bar{a}pR$ for rewarding the agent is higher when $\bar{a}$ is larger. When the project is close to its termination, $V' > 0$, rewarding the agent also comes with an indirect cost $\bar{a}pRV'$, because (27) shows that rewarding the agent decreases the agent's continuation utility, which pushes the agent's continuation utility closer to the termination boundary $\underline{U}$, thereby increasing the probability of costly termination. Meanwhile, volatility $l + U - pR$ of the agent's continuation utility is fixed for the same R. Therefore, the expected cost of offering a milestone bonus is larger when $\bar{a}$ is higher. Consequently, the principal reduces the milestone bonus and substitutes it with monitoring.

Panel (c) of Figure 4 shows that the principal monitors projects with larger payoffs more intensely, chooses a stronger contract sensitivity, and offers the agent more financial flexibility. This gives the agent more freedom to deliver big prize success. When project termination is more costly to the principal (lower L), Panel (d) shows that the principal rewards the agent less and offers a longer credit line. Again, a smaller milestone bonus helps to mitigate termination risk. A longer credit line also defers the more costly termination.

Multiple-stage project

In this section, we examine a two-stage project. Following the discussion around the baseline model, we consider the first-stage problem (19) after the second-stage problem is solved.

Similar to the argument leading to (29)–(31), we start with the following HJB equation for $V^{(1)}$:

$$(r+\bar{a})V^{(1)} = \max_{U^{(2)},R^{(1)},j} \Big\{ \bar{a}p^{(1)} \left[V^{(2)}(U^{(2)}) - R^{(1)} \right] + \left[(r+\bar{a})U - \bar{a}p^{(1)} \left(U^{(2)} + R^{(1)} \right) \right] \left(V^{(1)}\right)' + \frac{1}{2} j^2 s^2 \left(V^{(1)}\right)'' \Big\}, \tag{34}$$

and the boundary conditions

$$V^{(1)}(\underline{U}) = L, \quad \left(V^{(1)}\right)'(\overline{U}) = -1, \quad \text{and} \quad \left(V^{(1)}\right)''(\overline{U}) = 0, \tag{35}$$

for some $\overline{U}$.

The following comparison between the principal's values and the agent's equity stakes still hold in the current setting.

Proposition 4. *Suppose that* (9) *and* (10) *hold. Then* $V^{(1)}(U) < V^{(2)}(U)$ *for* $\underline{U} < U$ $\min \ \overline{U}^{(1)}, \overline{U}^{(2)}$ *. Therefore, the agent's equity stake* $\frac{U}{U+V^{(1)}(U)}$ *in the first stage is always larger than the equity stake* $\frac{U}{U+V^{(2)}(U)}$ *in the second stage.*

Figure 5 illustrates the optimal contract for a two-stage project. Similar to the baseline case, Panels (a) and (b) show that the principal's first-stage value is less than the second-stage value. Consequently, the agent's equity stake is higher in the first stage than in the second stage.

Due to positive contract sensitivity, the agent's continuation utility U is positively correlated with the signal about the agent's action. When positive evidence on the agent's action accumulates, the agent's continuation utility increases. Then, as Panel (b) of Figure 5 shows, his equity stake increases as well. This result is consistent with the empirical observations in Kaplan and Strömberg (2003) that a VC's equity stake is lower when an entrepreneur performs well compared to the bad performance state; hence, the entrepreneur's equity stake is higher in the good performance state. Moreover, (Kaplan and Strömberg, 2003, p. 287)also observe that "the state-contingencies are significantly greater in first VC rounds compared to subsequent ones." This finding is also consistent with our model prediction in Panel (b) that the derivative of the black solid line (the agent's equity stake in the first stage) is larger than the derivative of the red dotted line (the agent's equity stake in the second stage).

Panel (c) of Figure 5 shows the optimal $U^{(2)}$ as a function of the agent's continuation utility U at the time when the first stage is successfully completed. This panel shows that the agent receives a bump in his continuation utility from U to $U^{(2)}(U)$ when the first stage is successfully completed. Moreover, $U^{(2)}$ increases with U until U reaches a threshold and $U^{(2)}$ is bound at $\overline{U}^{(2)}$. Then, the first-stage milestone bonus turns positive and increases with U, as shown by the black solid line in Panel (e). In this case, the agent is rewarded with a maximum continuation utility $\overline{U}^{(2)}$ and also a milestone bonus $R^{(1)}$ to start the second stage of the project.

In the contract implementation discussed in Section 4, the maximum level of U is interpreted as the maximum balance of a credit line. We denote $\overline{U}^{(1)}$ as the maximum level of U in the first stage. $\overline{U}^{(1)}$ U represents the balance of the credit line in the first stage. For the example in Figure 5, $\overline{U}^{(1)}$ and $\overline{U}^{(2)}$ are approximately the same. Then, the bump from U to $U^{(2)}$ implies a reduction in the credit line balance to start the second-stage contract. In particular, when $U^{(2)}$ reaches $\overline{U}^{(2)}$, the agent's credit line balance is cleared before the second stage is initiated.

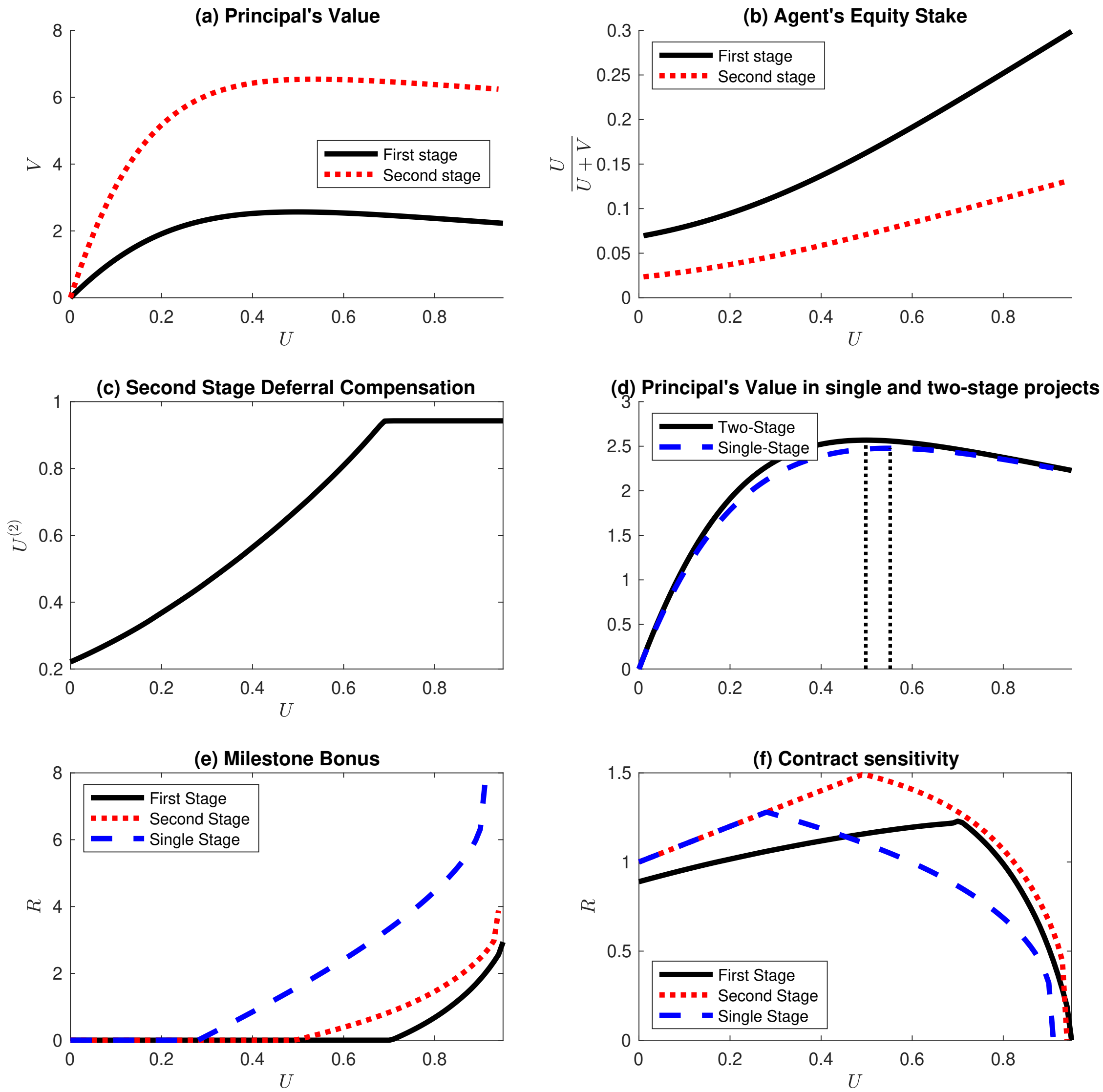


Figure 5. **Optimal contract for a two-stage project with a signal**

Notes: This figure plots the principal's value, the agent's equity stake, the agent's continuation utility $U^{(2)}$ *at the beginning of the second stage, milestone bonuses, and contract sensitivities for a two-stage project, and compares them with those for a single-stage project. The parameters used are* $r = 0.1$, $r = 0.05$, $s = 0.2$, $\bar{a} = 0.5$, $I = 1$, $D = 16$, $\underline{U} = 0$, *and* $L = 0$. *For the two-stage project, the success probabilities are* $p^{(1)} = p^{(2)} = 0.5$, *and the agent's maximum effort is* $\bar{a} = 0.5$ *in both stages. For the single-stage project, the success probability is* 0.25, *which matches the probability of final success in the two-stage project. The agent's maximum effort is* $\bar{a} = 0.5$ *in both stages of the two-stage project. The agent's maximum effort is* $\bar{a} = 0.25$ *in the single-stage project, so the expected success time in the two projects are the same.*

Panel (e) of Figure 5 shows the milestone bonuses at the end of the first stage (black solid line) and at the end of the second stage (red dotted line). Panel (f) presents the optimal contract sensitivities. In the first stage, when U is low, $R^{(1)} = 0$. However, the expected value of $U^{(2)}$ partially substitutes for monitoring so that $j_t < l + U_t$. When U reaches a threshold where $U^{(2)}$ reaches $\overline{U}^{(2)}$, if the first stage is still not completed, monitoring is further substituted with the milestone bonus $R^{(1)}$ and the contract sensitivity decreases to zero as U approaches $\overline{U}^{(1)}$. In the second stage of the project, when $R^{(2)} = 0$, the contract sensitivity (red dotted line in Panel (f)) is bound at $l + U_t$, until U reaches a threshold. Then, the monitoring is gradually phased out by the second-stage milestone bonus $R^{(2)}$.

Figure 5 also compares the principal's value and the optimal contract between a two-stage project and a single-stage project. For the single-stage project, the project either succeeds or fails at its single completion time. Therefore, the principal cannot terminate the project at an intermediate time when the first stage of the project fails. When both projects have the same success probability and the same expected final completion time, Panel (d) of Figure 5 shows that the principal's initial value for the two-stage project (the highest point of the black solid line) is higher than the initial value for the single-stage project (the highest point of the blue dashed line). This is because the two-stage project gives the principal the flexibility to terminate a failed project early instead of waiting until the final completion. As for the optimal contract, Panels (e) and (f) of Figure 5 show that the principal relies more on reward than monitoring to incentivize the agent's effort.

4 Capital structure implementation

In this section, we show how the optimal contract can be implemented using standard securities. The project raises initial funding by issuing the securities at time 0:

- **Long-Term Debt.** The project is initially funded by a general partner (principal) and a group of outside long-term debt holders. Debt holders receive continuous coupons at rate x. Let the borrowing rate be r so that the face value of the debt is $D = x/r$.

- **Credit Line.** A credit line provides the project available credit up to a limit $\overline{M}^{(i)}$ in stage i of the project. If the balance on the credit line reaches $\overline{M}^{(i)}$ in stage i, the project is terminated. When the credit line is fully repaid, all excess cash is paid to the agent. Balances on the credit line are charged at a fixed interest rate r.

- **Inside Equity.** The general partner offers the agent inside equity for the project. If stage i succeeds at time t, the inside equity held by the agent pays out cash dividend $R_t^{(i)}$ to the agent.

Proposition 5. *Consider a capital structure in which the general partner contributes at rate* $j_t dy_t$ *to the credit line according to her observation* y_t *of the signal. In stage* i*, a credit line with limit* $\overline{M}^{(i)} = \overline{U}^{(i)} \quad \underline{U}$ *is set up. The face value of the debt satisfies*

$$rD = \bar{a}\, l \quad r\overline{U}.[8]$$

The general partner offers the agent inside equity for the project. This inside equity rewards the agent cash dividend $R_t^{(i)}$ *if stage* i *succeeds at time* t*. At the same time, the agent receives a new credit line with limit* $\overline{M}^{(i+1)}$ *and initial balance* $\overline{U}^{(i+1)} \quad U^{(i+1)}$.

Under this capital structure, the agent's expected future payoff U_t *is determined by the current draw on the credit line*

$$U_t = \underline{U} + (\overline{M}^{(i)} \quad M_t)$$

when the project is in stage i*. This capital structure implements the optimal contract, and the agent exerts his maximum effort.*

5 Extensions

Two extensions of our model are presented in this section. In Section 5.1, we build on the baseline model from Section 2 by presenting a model in which the project payoff is uncertain and the principal can exercise a costly real option to learn the project payoff. The project is naturally broken into two stages: before and after the option exercise. We show that the option exercise resolves the uncertainty about project payoff; hence, the agent's equity stake is still smaller in the second stage. The pecking order between deferred compensation and milestone bonuses remains. As an extension of Section 3, we consider in Section 5.2 a model in which the principal can choose how precise the signal about the agent's effort is. We show that when negative (positive) evidence about the agent's effort accumulates, the principal increases (decreases) the signal precision so that monitoring intensifies (abates).

5.1 Information acquisition about the project payoff

So far we have assumed that the project payoff D is known to the principal. In this section, we consider an extension in which the project payoff is uncertain and could take binary values $D^l = 0$ or $D^h > 0$. The principal's prior distribution on D is $P(D = D^l) = 1 \quad q$ and $P(D = D^h) = q$. To resolve the uncertainty, the principal can exercise a costly real option

[8] If $x = \bar{a}\, l \quad r\overline{U} < 0$, the general partner also contributes to the credit line at a constant rate $\quad x$. In all our numeric experiments, x is positive.

to investigate the true value of the project. Depending on the investigation's findings, the principal either terminates the project early if the true payoff turns out to be of the low type, or continues the project if the payoff is of the high type. If the principal chooses to exercise the option, the project is naturally divided into two stages: before the exercise and after the exercise. In contrast to the previous models, completion of the first stage is determined by the principal in the current setting. If the project is of the high type, the principal offers the agent a new contract with a continuation utility U^{ex} at the beginning of the second stage. The principal can also award the agent a milestone bonus R^{ex} after the option exercise. If the principal chooses not to exercise the option, there is only one stage in the project, similar to the single-stage project in the baseline model. We will show that the pecking order between deferred compensation and milestone bonuses remains the same in the current setting.

If the project is completed at time n before the option exercise or termination, the agent's payoff depends on the project type. If a high type project succeeds at its completion, the agent receives a milestone bonus $R^{(1)}$. Otherwise, if either the project fails or the project is of low type, the agent receives nothing at the completion. Therefore, when the project is completed before the option exercise, the agent's expected reward right before completion is $p(a_n)qR_n^{(1)}$. Meanwhile, when the agent exerts his maximum effort, the principal's expected payoff, after netting rewards to the agent, is $pq(D^h - R_n^{(1)})$.

If the principal exercises the real option at time i to investigate the quality of the project, after the investigation, the agent's continuation utility depends on the revealed project type. If the project is of the high type, the principal offers the agent a new contract that provides the agent a continuation utility of U^{ex} for the next phase of the project. The principal's value in the next phase is

$$\begin{aligned} V_i^{(2)} &= V^{(2)}(U^{ex}) \\ &= \sup_{C,R^{(2)}} E_i^{\bar{a}}\left[\int_i^{t\wedge n} e^{-r(s-i)}(-dC_s) + 1_{\{n<t\}} e^{-r(n-i)} p(D^h - R_n^{(2)}) + 1_{\{t<n\}} e^{-r(t-i)} L\right], \end{aligned} \tag{36}$$

where there is no uncertainty about the project's payoff D^h. To illustrate the main mechanism, we assume that the principal does not observe the signal about the agent's effort and, therefore, can only incentivize the agent using C and milestone bonus $R^{(2)}$. Moreover, the principal can also award the agent a milestone bonus R^{ex} after the high type is revealed. If the project is of the low type, the principal terminates the project. Neither the agent nor the principal receive anything at termination.

Investigation is costly, and its cost to principal is k. Summarizing the previous two scenarios, the expected value from investigation is

$$G(U) = \sup_{U^{ex}, R^{ex}} q\left(V^{(2)}(U^{ex}) - R^{ex}\right) - k, \tag{37}$$

subject to

$$q(U^{ex} + R^{ex}) = U. \tag{38}$$

The constraint is a promise-keeping condition that the agent's continuation utility U right before the option exercise is exactly the expected payoff afterward. We call G the real option's exercise value.

When we introduce a Lagrangian multiplier h for the constraint (38), problem (37) is transformed into

$$\sup_{U^{ex}, R^{ex}} \left\{ q\left(V^{(2)}(U^{ex}) - R^{ex}\right) - k + hq\left(U^{ex} + R^{ex}\right) \right\}.$$

The first-order conditions with respect to U^{ex} and R^{ex} are

$$\left(V^{(2)}\right)'(U^{ex}) + h = 0 \quad \text{and} \quad h = 1.$$

Because $(V^{(2)})'(U^{ex}) > -1$ when U^{ex} is less than its maximum value $\overline{U}^{(2)}$, the previous two first-order conditions cannot hold simultaneously. The pecking order between deferred compensation and milestone bonuses remains valid: Only when $U^{ex} = \overline{U}^{(2)}$, does the principal award the agent a positive milestone bonus R^{ex} after the option exercise.

We formulate the optimal stopping problem facing the principal to determine the optimal investigation time in Appendix B. We present the results here.

Panel (a) in Figure 6 presents the principal's value before the option exercise. The red dotted line is the option's exercise value G, and the black solid line is the principal's value before the option exercise.[9] The project is initiated at the red asterisk on the black solid line, where the maximum value is achieved for the principal. After time 0, the agent's continuation utility U moves from 1.17 to the left. When U reaches 0.69 (red circle on the black solid line) before the project completion, it is optimal for the principal to exercise the real option. If the investigation reveals that the project is of the high type, the principal's value jumps from the red circle on the black solid line in Panel (b) to the red circle on the blue dashed line, which represents the principal's value $V^{(2)}$ in the second stage. The red circle on the blue dashed line represents $(U^{ex}, V^{(2)}(U^{ex}))$ with $U^{ex} = 1.39$. The second-stage of the project starts from

[9]The value function does not satisfy the smooth-pasting property due to the lack of Brownian noise. However, the value function satisfies the value-matching property between the value function and the exercise value.

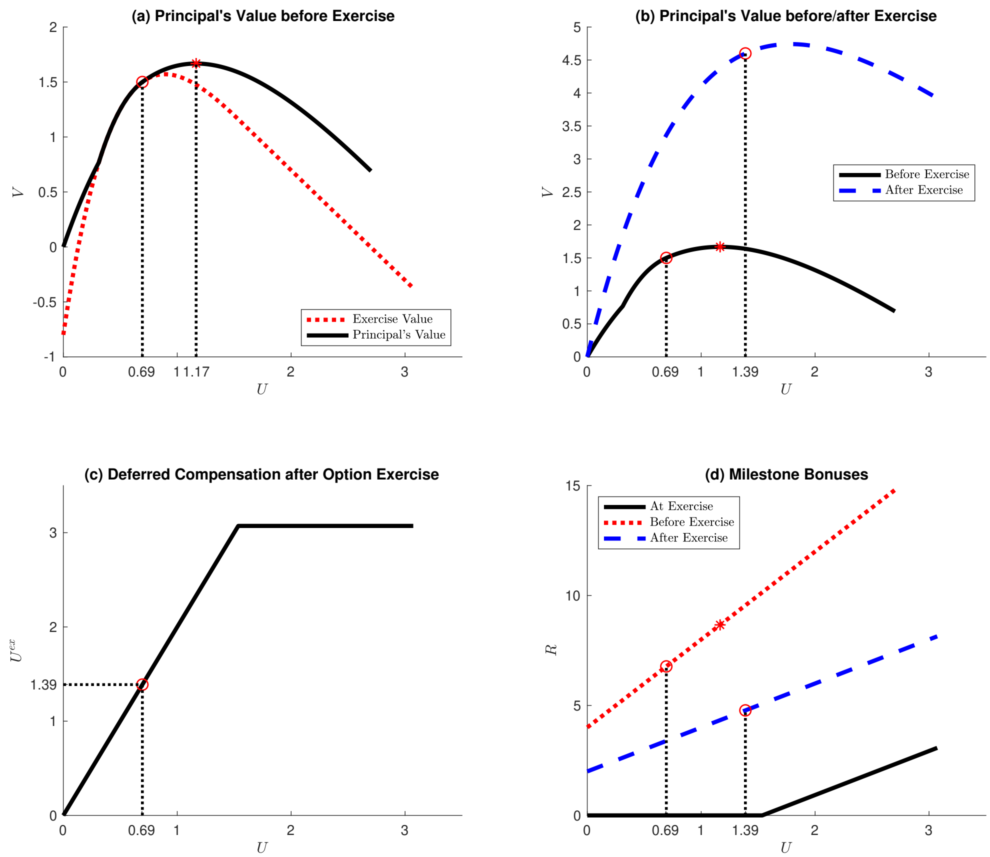


Figure 6. **Optimal contract with a real option**

Notes: This figure presents the principal's value before and after option exercise, deferred compensation at the beginning of the second stage, and the optimal milestone bonus. The parameters used are $r = 0.1$, $r = 0.05$, $\bar{a} = 0.5$, $I = 1$, $p = 0.5$, $q = 0.5$, $k = 0.8$, $D^h = 16$, $D^l = 0$, $\underline{U} = 0$, *and* $L = 0$.

this red circle and moves to the left. The project is either completed or terminated when U reaches zero.

Panel (b) of Figure 6 also shows that the principal's value after revealing the high type is larger than the value before. This change occurs because uncertainty because the project type has been resolved. A higher value after investigation implies that when the agent's continuation utility is the same, the equity stake is lower in the second stage than in the first.

Panel (c) of Figure 6 plots U^{ex} as a function of U. When U reaches 0.69, the principal exercises the option and awards the agent a new contract with continuation utility $U^{ex} = 1.39$. The black solid line in Panel (d) plots R^{ex} as a function of U. When $U = 0.69$, U^{ex} does not reach its maximum; hence, $R^{ex} = 0$. In other words, the principal does not award a milestone bonus after the option exercise in this case.

In addition to the black solid line, Panel (d) of Figure 6 also plots the milestone bonuses

$R^{(1)}$ and $R^{(2)}$, which the agent receives when the project succeeds, respectively, before and after the option exercise. In particular, after the project starts at the red asterisk on the red dotted line, as U moves to the left, the magnitude of $R^{(1)}$ decreases over time. If the project is completed before U reaches 0.69 and the project turns out to be of the high type, the red dotted line represents the milestone bonus $R^{(1)}(U)$ rewarded to the agent. If U reaches 0.69 but the project is not yet completed, the magnitude of the milestone bonus jumps from the red circle on the red dotted line to the red circle on the blue dashed line. The red circle on the blue dashed line represents $(U^{ex}, R^{(2)}(U^{ex}))$. After the option exercise, the magnitude of milestone bonus $R^{(2)}$ is represented by the blue dashed line. In the second stage, as U decreases, $R^{(2)}$ decreases as well.

Our model generates a pattern of success rewards that is decreasing in time in the first stage, jumps down after the high project type is revealed by the investigation, and decreases further in the second stage. Moreover, comparing the red dotted line and the blue dashed line in Panel (d) of Figure 6 , $R^{(1)}$ is higher than $R^{(2)}$ for the same level of U. Therefore, when the agent has the same amount of skin in the project, which is measured by U, the reward for success is larger in the first stage than in the second stage. The intuition for this result is as follows: Before the investigation, the project type is uncertain. Only when a high-type project is successfully completed does the agent receives the bonus $R^{(1)}$. Therefore, the agent's expected reward is $pqR^{(1)}$. However, after the investigation reveals that the project is of the high type, the agent is more certain to get the reward, so the expected reward after investigation changes to $pR^{(2)}$. Therefore, $R^{(1)}$ must be larger than $R^{(2)}$ to reach the same expected reward to balance the marginal cost $I + U$ and incentivize the agent. As a result, the success reward is larger when there is more uncertainty.

5.2 Optimal signal precision

Results in Section 3 show that the optimal contract depends less on the signal (lower contract sensitivity) as positive evidence of the agent's effort accumulates and the agent's continuation utility U increases. Therefore, the signal becomes less important to the principal as U increases. One would intuitively expect the principal to prefer a highly precise signal (close monitoring) when U is low, but can accept a low precision signal (loose monitoring) when U is high. We confirm this intuition by considering a model in which the principal can control the precision of the signal. In this model, we show that monitoring intensifies as negative evidence about the agent's effort accumulates.

The volatility s in (24) measures the information quality of the signal. The principal decides how closely to monitor the agent by choosing s at each time from a finite interval

$[s_{min}, s_{max}]$. A small value for s represents close monitoring by the principal, resulting in a more informative signal about the agent's effort. However, monitoring is costly to the principal. We assume that the monitoring cost per unit of time is

$$c(s) = \frac{M}{s^2} \tag{39}$$

for a positive constant M, such that the monitoring cost is proportional to the precision of the signal. We consider a single-stage project to illustrate the idea.

In this case, the principal's value satisfies the HJB equation

$$(r + \bar{a})V = \sup_{R, j, s} \left\{ \bar{a}p(D - R) - \frac{M}{s^2} + \left((r + \bar{a})U - \bar{a}pR\right) V' + \tfrac{1}{2} j^2 s^2 V'' \right\}, \tag{40}$$

subject to $R \in [0, D]$, $j \geq I + U - pR$, and boundary conditions (29) and (30).

The following proposition summarizes the principal's optimal signal choice, together with the optimal success reward and contract sensitivity.

Proposition 6. *When the principal can choose $s \in [s_{min}, s_{max}]$, subject to the monitoring cost in (39), all statements in Proposition 3 hold, with s in (32) replaced by the optimal s^*.*

When R^* is interior optimal, the optimal signal volatility is

$$s^* = \begin{cases} s_{min}, & -\frac{\bar{a}^2(1+V')^2}{2V''} > M \\ s_{max}, & -\frac{\bar{a}^2(1+V')^2}{2V''} \leq M \end{cases}, \tag{41}$$

and $s^* = s_{max}$ is in the neighborhood of $\overline{U}$.

To understand the optimal signal volatility choice, we plug the interior optimal $R^* = \frac{1}{p}\left(I + U + \frac{\bar{a}(1+V')}{s^2 V''}\right)$ into the right-hand side of (40) to obtain the following optimization in the signal precision $1/s^2$:

$$\max_{s \in [s_{min}, s_{max}]} \left(-\frac{\bar{a}^2(1 + V')^2}{2V''} - M \right) \frac{1}{s^2}.$$

Here, M is the marginal cost of signal precision. Because $V'' < 0$, the term $-\frac{\bar{a}^2(1+V')^2}{2V''}$ is positive, and we interpret it as the marginal benefit of signal precision. When the marginal benefit is larger than the marginal cost, the principal chooses the highest signal precision with $s^* = s_{min}$; otherwise, the principal chooses the lowest precision with $s^* = s_{max}$. When R^* is binding at 0 or D, the optimal s^* could be interior.

Solving the HJB equation (40) together with boundary conditions (29) and (30) numerically, we present the optimal contract and the optimal signal precision choice in

Figure 7.

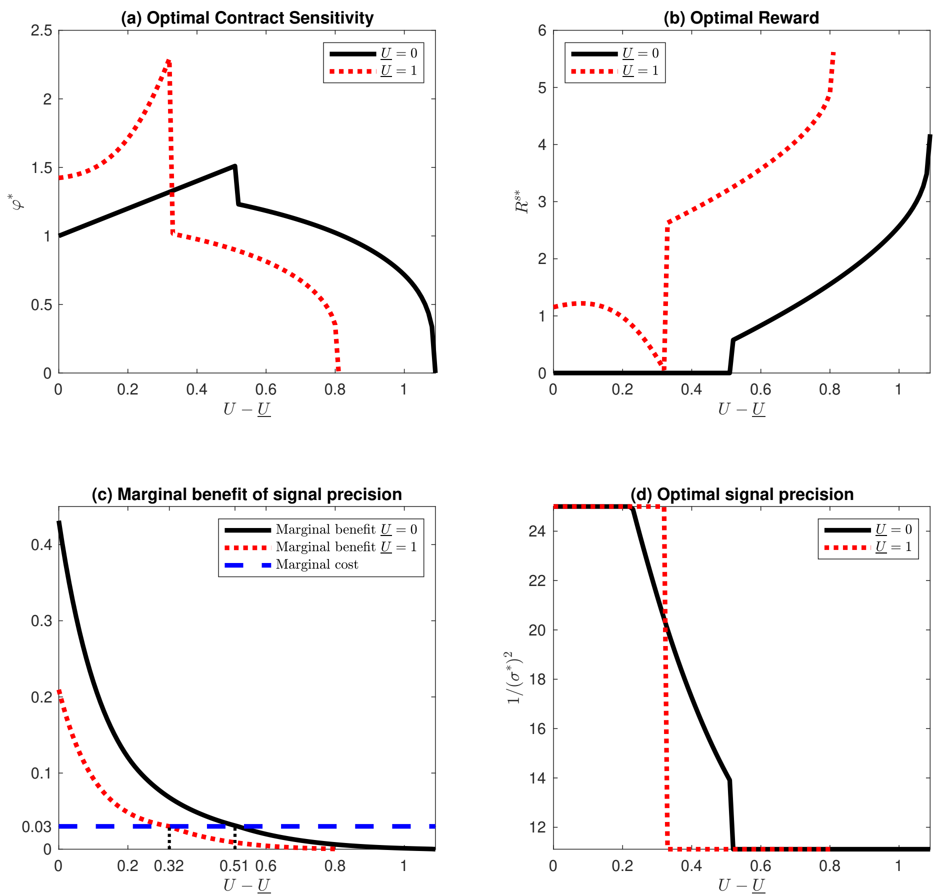


Figure 7. **Optimal contract and signal precision choice**

Notes: This figure plots the optimal contract sensitivities, the milestone bonuses, the marginal benefit of signal precision, and the optimal signal precision when the agent's outside value $\underline{U}$ *is either* 0 *or* 1*. Other parameters used are* $r = 0.1$, $r = 0.05$, $\bar{a} = 0.5$, $l = 1$, $s_{min} = 0.2$, $s_{max} = 0.3$, $M = 0.03$, $p = 0.5$, *and* $L = 0$.

Panel (c) in Figure 7 shows that the marginal benefit of signal precision decreases as the agent's continuation utility U increases, and it converges to zero as U approaches the payment boundary $\overline{U}$. As a result, the principal chooses high signal precision and monitors the agent closely when the project is close to termination, but she chooses low signal precision and loosens monitoring when the agent has sufficient skin in the project. When the agent's outside value $\underline{U}$ is zero and the agent's continuation utility U is small, Panel (b) shows that the milestone bonus R is bound at zero. This avoids a negative drift $(r + \bar{a})U \quad \bar{a}R$ of the agent's continuation utility and mitigates the risk of project termination. When $\underline{U}$ is larger, the principal can choose a positive milestone bonus R to simultaneously avoid negative drift and reduce the volatility $l + U \quad pR$ for the agent's continuation utility. Hence, the red dotted line in Panel (b) shows $R > 0$ when U is close to $\underline{U}$. In both cases, R jumps when the principal switches to the lowest signal precision. In this regime, R increases to $(I + \overline{U})/p$ and the optimal contract sensitivity j decreases to zero, as U approaches the payment boundary $\overline{U}$. When R is not bound by either 0 or D, the red dotted line in Panel (d) shows that the principal either chooses the highest signal precision when the agent's continuation utility is low or the lowest signal precision when the continuation utility is close to the payment boundary, confirming the result in Proposition 6.

6 Model empirical predictions

Several predictions of our model are supported by the empirical literature on venture capital financing. Our model predicts that the agent's equity stake is smaller in later financing rounds. This prediction is due to the resolution of uncertainty in later stages, which increases the project value for the principal and decreases the entrepreneur's equity stake. Project uncertainty can be either its success probability or its potential payoffs. Project success is more certain in later stages because early stages are successfully completed (see the baseline model in Section 2 and the model with monitoring in Section 3) and because the project payoff is more certain in later stages after investigation (see the model in Section 5.1). This prediction is consistent with Bengtsson and Sensoy (2015) and Ewens, Nanda, and Stanton (2020), who document that an entrepreneur's equity stake is lower in later funding rounds and decreases with firm age.

When the principal observes a signal about the agent's effort, our model illustrates that the reward for success increases with positive evidence about the agent's effort and that the state-dependence is more significant in the first VC rounds compared to subsequent ones. These results are consistent with the empirical evidence of Kaplan and Strömberg (2003).

Our model generates two additional predictions that could be interesting for empirical testing.

1. **Pecking order between milestone bonuses and deferred compensation.** Our model shows that when a project succeeds in its intermediate stage, it is more efficient to reward the agent deferred compensation rather than cash. This result implies a back-loaded cash reward for entrepreneurs.

   This prediction is related to Ewens, Nanda, and Stanton (2020). They show that having a "product–market fit" is a significant milestone at which point cash compensation increases significantly for CEOs. They argue that having product–market fit reduces the reliance on key human capital and that the threat of inefficient project termination due to the departure of key personnel becomes less severe.

Oyer (2004) and Oyer and Schaefer (2005) find empirical evidence that firms generally prefer to grant deferred compensation (e.g., stock options) to its employees rather than paying a cash bonus, specifically to improve retention, which is consistent with our results. To test and measure the threshold strategy, one could implement automatic manipulation tests based on density discontinuity (see McCrary (2008) and Cattaneo, Jansson, and Ma (2018, 2020)) using firm-level data from the National Center for Employee Ownership (NCEO), firm annual reports, and data on the incidence of employee stock options from the Bureau of Labor Statistics Pilot Survey.

2. **Complementarity between monitoring and reward.** Our model shows that monitoring and rewarding success complement each other in the optimal contract. When positive (negative) evidence about the agent's effort accumulates, the agent's compensation depends less (more) on the documented evidence, the reward for project success increases (decreases), and the principal loosens (intensifies) monitoring.

   Strict monitoring in bad states relates to board supervision in Kaplan and Strömberg (2003). The authors show that when performance is lower than expected, the VC gets more board control rights and voting control. As performance improves, the entrepreneur retains more control rights, whereas VCs retain their cash-flow rights but relinquish most of their control rights.

   To analyze the relationship between monitoring and reward, one potential strategy is to use the data set in Bernstein, Giroud, and Townsend (2016). They show that the introduction of new airline routes that reduce VCs' travel times to their existing portfolio companies increase VCs' involvement and increase both innovation and the likelihood of a successful exit. It would be interesting to examine VC contracts with their portfolio companies to determine whether the potential rewards (such as entrepreneur cash flow rights or equity stakes) are reduced due to more VC involvement after the introduction of new airline routes.

7 Conclusion

This paper presents a dynamic principal-agent model for financing a project that has three salient features of a long-term research project or a venture capital–backed startup. First, the project has multiple stages and only yields a final payoff when all stages are successfully completed. Second, the agent's effort and involvement in the project are hard to observe or monitor. Third, key project personnel have a limited commitment of human capital, and the project is inefficiently terminated once the key personnel leave the project.

The optimal contract has several unique features: (1) It is suboptimal to reward the agent with cash bonuses for intermediate-stage success unless the agent's continuation utility exceeds a threshold. Hence, there is a pecking order between deferred compensation and cash bonuses. Deferred compensation should be used first to reward milestone success, and cash bonuses should only rewarded when the deferred compensation reaches its maximum. (2) The agent's equity stake in the project is positively related to performance and is also lower and less sensitive to performance in later stages than early ones. This is due to the resolution of project uncertainties in later stages. Because a project is closer to its final success and its payoff is more certain in later stages, the project value is larger in later stages, resulting a smaller equity stake for the agent. (3) When the agent's effort can be imperfectly monitored, the incentive provision of monitoring and reward dynamically complement each other in the optimal contract. When negative (positive) evidence about the agent's effort accumulates, monitoring intensifies (abates) and the reward is reduced (increased). We link these model predictions to empirical evidence from venture capital contracts and propose additional empirical tests.

It would be interesting to incorporate additional features into our model setting. For example, the model could incorporate the long-term impacts of the agent's actions. For long-term research projects, trials and experiments in the past, even if they fail, can provide valuable information for the project and can ultimately lead to project success. Therefore, it is realistic to assume that the agent's effort has long-term impacts. Understanding the optimal contract in this setting is challenging. We leave it for future research.

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Appendix A Proofs

Lemma 2 is proved below. Lemma 1 is a special example. In this proof, we consider the possibility that principal pays agent a severance pay R^f if a stage fails. We show that the expected payoff at a stage completion incentivizes the agent, rather than the success reward or severance pay individually.

Proof of Lemma 2

We first use the following identities to transform the agent's objective function in (5):

$$\mathbb{P}^a\left(n^{(i)} > s \mid n^{(i)} > t\right) = e^{-\int_t^s a_u du} \quad \text{and} \quad \mathbb{P}^a\left(n^{(i)} \in [s, s+ds) \mid n^{(i)} > t\right) = a_s ds\, e^{-\int_t^s a_u du}. \tag{A.1}$$

The later identity holds because the conditional probability $\mathbb{P}^a(n^{(i)} \in [s, s+ds) \mid n^{(i)} > t)$ is the product of survival probability $\mathbb{P}^a(n^{(i)} > s \mid n^{(i)} > t) = e^{-\int_t^s a_u ds}$ and the probability $\mathbb{P}^a(n^{(i)} \in [s, s+ds) \mid n^{(i)} > s) = a_s ds$. (See Jacod (1975) for a rigorous treatment.) Utilizing the two identities in (A.1), the agent's objective function in (5) is transformed to

$$\begin{aligned}
&\mathbb{E}_t^a\left[\int_t^t 1_{\{n^{(i)}>s\}} e^{-r(s-t)}\left(dC_s + l(\bar{a} - a_s)ds\right) + \int_t^t 1_{\{n^{(i)} \in [s,s+ds)\}} e^{-r(s-t)} U_s^c ds + 1_{\{t<n^{(i)}\}} e^{-r(t-t)} \underline{U}\right] \\
&= \mathbb{E}_t^a\left[\int_t^t e^{-\int_t^s a_u du} e^{-r(s-t)}\left(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\right) + e^{-\int_t^t a_u du} e^{-r(t-t)} \underline{U}\right].
\end{aligned} \tag{A.2}$$

Consider the problem in the first stage. The proof for later stages is similar. A contract $(C, U^{(2)}, R^{(1)}, R^f)$ is admissible if it satisfies the integrability assumption

$$\mathbb{E}^a\left[\int_0^\infty e^{-rs}\left(dC_s + a_s U_s^c ds\right)\right] < \infty,$$

for any process a taking values in $[0, \bar{a}]$ and $U^{(2)} + R^{(1)} > R^f$. Here $U_t^c = p^{(1)}(a_t)(U_t^{(2)} + R_t^{(1)}) + (1 - p^{(1)}(a_t))R^f$. The assumption $U^{(2)} + R^{(1)} > R^f$ ensures reward for success is larger than failure so that the agent does not strategically explore project failure, which is not the best interest for principal.

Given an admissible contract and any agent's effort a, define its associated value U_t^a as the right-hand side of (A.2). The admissibility of the contract implies that $e^{-\int_0^t a_u du} e^{-rt} U_t^a + \int_0^t e^{-\int_0^s a_u du} e^{-rs}\left(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\right)$ is a $\mathbb{P}^a$-martingale. The martingale representation theorem implies the existence of j^a such that U^a follows the dynamics

$$
\begin{aligned}
dU_t^a &= (r + a_t)U_t^a dt - l(\bar{a} - a_t)dt - a_t U_t^c dt + \phi^a_s dB_t^a - dC_t \\
&= rU_t^a dt + \Big\{ a_t U_t^a - l(\bar{a} - a_t) - a_t p^{(1)}(a_t)\big(U_t^{(2)} + R_t^{(1)} - R^f\big) - a_t R^f - \phi_t^a a_t \Big\} dt + \phi_t^a dy_t - dC_t,
\end{aligned}
$$

with the terminal condition $U_\tau^a = \underline{U} + C_\tau - C_{\tau-}$. Equations (5) and (A.2) combined yield $U_t = \sup_a U_t^a$. It then follows from the comparison theorem for BSDEs (cf. El Karoui, Peng, and Quenez (1997)) that U follows the dynamics

$$
\begin{aligned}
dU_t = rU_t dt &+ \inf_{a_t \in [0,\bar{a}]} \Big\{ a_t U_t - l(\bar{a} - a_t) - a_t p^{(1)}(a_t)\big(U_t^{(2)} + R_t^{(1)} - R^f\big) - a_t R^f - \phi_t a_t \Big\} dt \\
&+ \phi_t dy_t - dC_t.
\end{aligned}
\tag{A.3}
$$

This dynamics ensures that

$$
\Gamma_t = e^{-\int_0^t a_u du} e^{-rt} U_t + \int_0^t e^{-\int_0^s a_u du} e^{-rs} \big(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\big)
$$

satisfies the martingale principle: Γ is a $\mathbb{P}^a$-supermartingale for arbitrary a and is a $\mathbb{P}^{a^*}$-martingale for the optimal a^*. Because $a \mapsto ap(a)$ is convex and $U^{(2)} + R^{(1)} > R^f$, the optimizer of the minimization problem in (A.3) can only be achieved at $a = 0$ or $a = \bar{a}$. Comparing the values of these two actions, we obtain the optimal a^* as

$$
a_t^* = \begin{cases} \bar{a}, & U_t^c + \phi_t \geq l + U_t, \\ 0, & \text{otherwise.} \end{cases}
$$

This implies that agent's optimal effort is incentivized by the expected reward U^c, rather than $U^{(2)}$, $R^{(1)}$, and R^f individually.

To verify the optimality of a^*, we obtain from (A.3) implies that $dU_t \geq \big(rU_t + a_t U_t - l(\bar{a} - a_t) - a_t U_t^c - \phi_t a_t\big) dt + \phi_t dy_t - dC_t$ for arbitrary a and the inequality becomes an equality for a^*. Hence Γ_t is a $\mathbb{P}^a$-supermartingale and a $\mathbb{P}^{a^*}$-martingale when a is replaced by a^*. As a result,

$$
U_t \geq \mathbb{E}_t^a \Big[\int_t^\tau e^{-\int_t^s a_u du} e^{-r(s-t)} \big(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\big) + e^{-\int_t^\tau a_u du} e^{-r(\tau - t)} \underline{U} \Big]
$$

and the inequality becomes an equity when a is replaced by a^*. This confirms the optimality of a^*. □

Due to the risk neutrality of principal, principal's value depends on the expected cash compensation $p^{(1)}R^{(1)} + (1 - p^{(1)})R^f$ at completion, rather than $R^{(1)}$ and R^f individually.

Therefore, because both the agent's continuation utility and the principal's value depend on the expected reward rather than cash compensation in different states, we set $R^f = 0$ and (A.3) is reduced to (25).

If the agent can choose any time $\tilde{\tau} \geq \tau$ to quit the project, we will show $\tilde{\tau} = \tau$ is agent's optimal stopping time. To this end, for arbitrary $\tilde{\tau} \geq \tau$, due to the $\mathbb{P}^a$-supermartingale property of Θ,

$$\begin{aligned} U_0 = \Theta_0 &\geq \mathbb{E}^a\left[\int_0^{\tilde{\tau}} e^{-\int_0^s a_u du} e^{-rs}\left(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\right) + e^{-\int_0^{\tilde{\tau}} a_u du} e^{-r\tilde{\tau}} U_{\tilde{\tau}}\right] \\ &\geq \mathbb{E}^a\left[\int_0^{\tilde{\tau}} e^{-\int_0^s a_u du} e^{-rs}\left(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\right) + e^{-\int_0^{\tilde{\tau}} a_u du} e^{-r\tilde{\tau}} \underline{U}\right], \end{aligned}$$

where the second inequality follows from $U_{\tilde{\tau}} \geq \underline{U}$. Both inequalities above are equalities when $a = a^*$ and $\tilde{\tau} = \tau$. This confirms that

$$U_0 = \sup_{a \in [0,\bar{a}],\, \tilde{\tau} \geq \tau} \mathbb{E}^a\left[\int_0^{\tilde{\tau}} e^{-\int_0^s a_u du} e^{-rs}\left(dC_s + l(\bar{a} - a_s)ds + a_s U_s^c ds\right) + e^{-\int_0^{\tilde{\tau}} a_u du} e^{-r\tilde{\tau}} \underline{U}\right]$$

and (a^*, τ) are the optimal effort and stopping time.

Proof of Proposition 1

We first derive the HJB equation (13). To this end, the same argument as in (A.2) transforms V into the following form

$$V_t = \sup_{R,C} \mathbb{E}_t^{\bar{a}}\left[\int_t^{\tau} e^{-r(s-t)} e^{-\int_t^s \bar{a} dz}\left(dC_s + \left(\bar{a}pD - R_n\right) ds\right) + e^{-r(\tau-t)} e^{-\int_t^{\tau} \bar{a} dz} L\right]. \quad \text{(A.4)}$$

Then (13) follows from the dynamic programming when $dC_t = c_t dt$.

When $U < \overline{U}$, $c = 0$, plugging (16) into (13), we obtain

$$1 + V' = \frac{\bar{a}pD - (r + \bar{a})U - (r + \bar{a})V}{\bar{a}l - rU}. \quad \text{(A.5)}$$

Due to assumptions (9) and (10), the right-hand side of the previous equation is larger than zero when $U = \underline{U}$, implies that $V'(\underline{U}) > -1$ and $\overline{U} > \underline{U}$.

We next prove that $\bar{a}pD - (r + \bar{a})U - (r + \bar{a})V > 0$ when $U < \overline{U}$. To this end, $V'(\overline{U}) = -1$ and (A.5) imply that

$$\bar{a}pD - (r + \bar{a})\overline{U} - (r + \bar{a})V(\overline{U}) = 0. \quad \text{(A.6)}$$

Starting from $\overline{U}$ and moving to the left, if there is a sequence $\{U_n\}_n$ converging to $\overline{U}$ such that

$(r+\bar{a})V(U_n) + (r+\bar{a})U_n \quad \bar{a}p\mathrm{D}$, then

$$V(U_n) \qquad \frac{\bar{a}p\mathrm{D} \quad (r+\bar{a})U_n}{r+\bar{a}}.$$

Combining the previous inequality with (A.6), we obtain

$$\frac{V(\overline{U}) \quad V(U_n)}{\overline{U} \quad U_n} \qquad \frac{r+\bar{a}}{r+\bar{a}},$$

which implies $V'(\overline{U}) \qquad \frac{r+\bar{a}}{r+\bar{a}} < \quad 1$ due to $r > r$. This contradicts with $V'(\overline{U}) = \quad 1$. Therefore, we confirm the claim.

Now plugging $1 + V' > 0$ and $\bar{a}p\mathrm{D} \quad (r+\bar{a})U \quad (r+\bar{a})V(U) > 0$ into (A.5), we conclude that $\bar{a}I \quad rU > 0$ when $U < \overline{U}$. Combining (8) with (16), we obtain the dynamics of U:

$$dU_t = (rU_t \quad I\bar{a})dt.$$

Therefore U decreases along its path because $rU \quad \bar{a}I < 0$.

Proof of Proposition 2

(i) When $U = p^{(1)}\overline{U}^{(2)} \quad I < \overline{U}^{(1)}$, in order to satisfy the incentive compatible constrain, we must have

$$p^{(1)}(U^{(2)} + R^{(1)}) \quad I + U = p^{(1)}\overline{U}^{(2)}. \tag{A.7}$$

Because $(V^{(1)})'(U) > \quad 1$ for $U < \overline{U}^{(1)}$, the first order condition of $R^{(1)}$ in (23) implies $R^{(1)} = 0$ and $U^{(2)} = \overline{U}^{(2)}$ in order to satisfy the inequality (A.7).

When $p^{(1)}\overline{U}^{(2)} \quad I < U < \overline{U}^{(1)}$, $U^{(2)}$ must be $\overline{U}^{(2)}$ to satisfy the incentive compatible constraint $p^{(1)}(U^{(2)} + R^{(1)}) \quad I + U$. Otherwise when $U < \overline{U}^{(2)}$, $R^{(1)} = 0$, the inequality $p^{(1)}(U^{(2)} + R^{(1)}) \quad I + U$ cannot be satisfied. Because $(V^{(1)})'(U) > \quad 1$, the first order condition of $R^{(1)}$ in (23) implies that the incentive compatible constraint must be binding, hence $R^{(1)} = \frac{1}{p^{(1)}}(I + U) \quad \overline{U}^{(2)}$.

(ii) We only need to show $U^{(2)} > U$ when the first stage succeeds and $U^{(2)} \quad \overline{U}^{(2)}$. In this case, $R^{(1)} = 0$, then $p^{(1)}U^{(2)} \quad I + U$ implies $U^{(2)} > U$ thanks to $p^{(1)} \quad 1$ and $I > 0$.

(iii) We claim that

$$p^{(1)} \; V^{(2)}(U^{(2)}) + U^{(2)} \; < p^{(2)}\mathrm{D}, \quad \text{for any } U^{(2)} \quad \overline{U}^{(2)}. \tag{A.8}$$

To prove this claim, recall from the proof of Proposition 1 that

$$(r+\bar{a})V^{(2)}(U^{(2)}) + (r+\bar{a})U^{(2)} \leq \bar{a}p^{(2)}D.$$

Therefore, this inequality and $r > r$ combined implies

$$p^{(1)}\left(V^{(2)}(U^{(2)}) + U^{(2)}\right) \leq p^{(1)}\frac{\bar{a}p^{(2)}D}{r+\bar{a}} < p^{(2)}D.$$

The claim is then confirmed.

Using (3) and (A.1), problem (19) is transformed into

$$\begin{aligned}
V_t^{(1)} &= \sup_{U^{(2)},R^{(1)},C} E_t^{\bar{a}}\left[\int_t^{\tau} e^{-(r+\bar{a})(s-t)}\left(dC_s + \bar{a}p^{(1)}\left(V^{(2)}(U_s^{(2)}) - R_s^{(1)}\right)ds\right) + e^{-(r+\bar{a})(\tau-t)}L\right] \\
&= \sup_{U^{(2)},R^{(1)},C} E_t^{\bar{a}}\left[\int_t^{\tau} e^{-(r+\bar{a})(s-t)}\left(dC_s + \bar{a}p^{(1)}\left(V^{(2)}(U_s^{(2)}) + U_s^{(2)}\right) - \bar{a}p^{(1)}\left(U_s^{(2)} + R_s^{(1)}\right)ds\right)\right. \\
&\qquad \left. + e^{-(r+\bar{a})(\tau-t)}L\right] \\
&< \sup_{U^{(2)},R^{(1)},C} E_t^{\bar{a}}\left[\int_t^{\tau} e^{-(r+\bar{a})(s-t)}\left(dC_s + \bar{a}p^{(2)}D - \bar{a}p^{(1)}(U_s^{(2)} + R_s^{(1)})ds\right) + e^{-(r+\bar{a})(\tau-t)}L\right],
\end{aligned}$$

where the inequality follows from (A.8). Introduce $R^{(2)}$ so that $p^{(2)}R_s^{(2)} = p^{(1)}(U_s^{(2)} + R_s^{(1)})$. The right-hand side of previous inequality is less than

$$\sup_{R^{(2)},C} E_t^{\bar{a}}\left[\int_t^{\tau} e^{-(r+\bar{a})(s-t)}\left(dC_s + \bar{a}p^{(2)}\left(D - R_s^{(2)}\right)ds\right) + e^{-(r+\bar{a})(\tau-t)}L\right],$$

which is exactly $V_t^{(2)}$.

Once we have $V^{(1)}(U) < V^{(2)}(U)$, the claim $\frac{U}{U+V^{(1)}(U)} > \frac{U}{U+V^{(2)}(U)}$ follows.

Proof of Propositions 3 and 6

We will prove the statements for the general case with nonnegative information cost in (39). Proposition 3 is a special case with $M = 0$ and $s_{min} = s_{max}$.

Concavity of V. The term with j is $\frac{1}{2}j^2s^2V''$ on the right-hand side of (40). Because j is unbounded, V'' must be nonpositive, otherwise the right-hand side of (40) is infinite, which contradicts with the finite V on the left-hand side.

To show the strict concavity of V, we start from $\overline{U}$, where $V'(\overline{U}) = -1$ and $V''(\overline{U}) = 0$.

Then the limit of (40) as $U \to \overline{U}$ is reduced to

$$(r+\bar{a})V(\overline{U}) = \sup_{s} \Big\{ \bar{a}pD - \tfrac{M}{s^2} - (r+\bar{a})\overline{U} \Big\} = \bar{a}pD - \tfrac{M}{s_{max}^2} - (r+\bar{a})\overline{U}.$$

Starting from $\overline{U}$ and moving to the left, it is necessary that $V(U) < \frac{1}{r+\bar{a}}\left(\bar{a}pD - \frac{M}{s_{max}^2} - (r+\bar{a})U\right)$ for U in a left neighborhood of $\overline{U}$. Otherwise, there exists a sequence of points $U_n \to \overline{U}$ such that $V(U_n) \geq \frac{1}{r+\bar{a}}\left(\bar{a}pD - \frac{M}{s_{max}^2} - (r+\bar{a})U_n\right)$. This would yield $V'(\overline{U}) \leq -\frac{r+\bar{a}}{r+\bar{a}} < -1$ because $r > r$ and contradicts with $V'(\overline{U}) = -1$. If $V(U) < \frac{1}{r+\bar{a}}\left(\bar{a}pD - \frac{M}{s_{max}^2} - (r+\bar{a})U\right)$ for U in a left neighborhood of $\overline{U}$, then $V'' < 0$ in the same neighborhood. To see this, suppose otherwise there is a U such that $V''(U) = 0$, then (40) is reduced to

$$\begin{aligned}(r+\bar{a})V &= \sup_{R,s} \Big\{ \bar{a}p(D-R) - \tfrac{M}{s^2} + \big((r+\bar{a})U - \bar{a}pR\big)V' \Big\} \\ &= \bar{a}pD - \tfrac{M}{s_{max}^2} + (r+\bar{a})UV' \\ &\geq \bar{a}D - \tfrac{M}{s_{max}^2} - (r+\bar{a})U,\end{aligned}$$

where the second equality follows because $1+V' \geq 0$, hence R maximizing $-\bar{a}R(1+V')$ is zero, the inequality holds due to $V' \geq -1$. However, this inequality contradicts with $V(U) < \frac{1}{r+\bar{a}}\left(\bar{a}D - \frac{M}{s_{max}^2} - (r+\bar{a})U\right)$. Once the concave function V is below the line $V(U) = \frac{1}{r+\bar{a}}\left(\bar{a}D - \frac{M}{s_{max}^2} - (r+\bar{a})U\right)$, it can never go above this line. We have just seen that V below this line implies the strict concavity of V, hence V is strictly concave on $(\underline{U}, \overline{U})$.

Optimal R **and** j. The concavity of V implies that the optimal j is $I+U-R$. The equation (40) is then reduced to

$$(r+\bar{a})V = \sup_{R,s} \Big\{ \bar{a}p(D-R) - \tfrac{M}{s^2} + \big((r+\bar{a})U - \bar{a}pR\big)V' + \frac{1}{2}s^2(I+U-R)^2V'' \Big\}. \tag{A.9}$$

For given s, the first order condition in R yields

$$R = \min\Big\{ \max\Big\{ \frac{1}{p}\Big(I+U+\frac{\bar{a}(1+V')}{s^2V''}\Big), 0\Big\}, D\Big\}.$$

Because $1+V' > 0$ and $V'' < 0$ when $U < \overline{U}$, the previous expression of R with $s = s$ implies that $pR < I+U$, hence $j > 0$.

When the optimal R is interior optimal, i.e., $R = \frac{1}{p}\left(I+U+\frac{\bar{a}(1+V')}{(s)^2V''}\right)$, plugging this expression into (40), we obtain

$$(r+\bar{a})V = \sup_{s\in[s_{min},s_{max}]} \Big\{ \bar{a}pD - \bar{a}(I+U) + V'(rU - \bar{a}I) - \Big(\frac{\bar{a}^2(1+V')^2}{2V''} + M\Big)\frac{1}{s^2} \Big\}. \tag{A.10}$$

Note that the optimization in $\frac{1}{s^2}$ is linear. Therefore the optimal s in (41) is confirmed.

Limit of R, j and s as $U \to \overline{U}$. We will first prove by contradiction that R cannot be zero in a neighborhood of $\overline{U}$. Suppose otherwise, then equation (A.9) is simplified to

$$(r + \bar{a})V = \sup_{s} \left\{ \bar{a}D \quad \frac{M}{s^2} + (r + \bar{a})UV' + \frac{1}{2}s^2(l + U)^2V'' \right\}, \tag{A.11}$$

when U is in a neighborhood of $\overline{U}$. Because $\lim_{U \to \overline{U}} V'' = 0$, $\frac{M}{s^2} + \frac{1}{2}s^2(l + U)^2V''$ is increasing in s when U is sufficiently close to $\overline{U}$. Therefore, we would have $s = s_{max}$ in a neighborhood of $\overline{U}$.

Taking derivative with respect to U on both sides of (A.11),

$$(r + \bar{a})V' = (r + \bar{a})V' + (r + \bar{a})UV'' + s_{max}^2(l + U)V'' + \frac{1}{2}s_{max}^2(l + U)^2V''',$$

when U is close to $\overline{U}$. Sending $U \to \overline{U}$ and using the boundary condition (29), we obtain from the previous equation

$$r \quad r = \frac{1}{2}s_{max}^2(l + \overline{U})V'''(\overline{U}).$$

Therefore, due to $r < r$, $V'''(\overline{U})$ is positive and finite. It then follows from L'Hôpital rule and (29) that

$$\lim_{U \to \overline{U}} \frac{1 + V'}{V''} = \lim_{U \to \overline{U}} \frac{V''}{V'''} = 0.$$

As a result, (32) implies that $\lim_{U \to \overline{U}} R = \min\{(l + \overline{U})/p, D\}$, which contracts with the assumption that R is zero in a neighborhood of $\overline{U}$.

Given that the constraint $R \quad 0$ is not binding in a neighborhood of $\overline{U}$, R is the minimal between D and $\frac{1}{p} \quad l + U + \frac{\bar{a}(1+V')}{s^2V''}$. In the later case, (A.10) implies that

$$(r + \bar{a})V = \bar{a}pD \quad \frac{M}{(s)^2} \quad \bar{a}(l + U) + (rU \quad \bar{a}l)V' \quad \frac{1}{2}\frac{\bar{a}^2(1 + V')^2}{(s)^2V''} \tag{A.12}$$

holds in a neighborhood of $\overline{U}$. Sending $U \to \overline{U}$ and using the boundary condition $V'(\overline{U}) = 1$, we obtain from the previous equation that

$$(r + \bar{a})V(\overline{U}) = \bar{a}pD \quad \frac{M}{(s)^2} \quad (r + \bar{a})\overline{U} \quad \lim_{U \to \overline{U}} \frac{1}{2}\frac{\bar{a}^2(1 + V')^2}{(s)^2V''}.$$

Meanwhile combining (29) and (40) yields

$$(r + \bar{a})V(\overline{U}) = \bar{a}pD \quad \frac{M}{(s)^2} \quad (r + \bar{a})\overline{U}.$$

Therefore, combining the previous two equations and using $s > 0$, we obtain

$$\lim_{U \to \overline{U}} \frac{(1+V')^2}{V''} = 0. \tag{A.13}$$

Therefore (41) implies $s = s_{max}$ in a neighborhood of $\overline{U}$.

Taking the derivative with respect to U on both sides of (A.12) with $s = s_{max}$, we obtain

$$(r+\bar{a})V' = -\bar{a} + rV' + (rU - \bar{a}I)V'' - \frac{\bar{a}^2}{2s_{max}^2}\Big[2(1+V') - \frac{(1+V')^2 V'''}{(V'')^2}\Big].$$

Sending $U \to \overline{U}$, we obtain

$$r - r = \lim_{U \to \overline{U}} \frac{\bar{a}^2}{2s_{max}^2} \frac{(1+V')^2}{(V'')^2} V'''. \tag{A.14}$$

Combining (A.13) and (A.14) and using the fact that $V'' < 0$ and $r > r$, we obtain

$$\lim_{U \to \overline{U}} \frac{V'''}{V''} = -\infty. \tag{A.15}$$

It then follows from L'Hôpital rule that

$$\lim_{U \to \overline{U}} \frac{1+V'}{V''} = \lim_{U \to \overline{U}} \frac{V''}{V'''} = 0.$$

As a result,

$$\lim_{U \to \overline{U}} R = \lim_{U \to \overline{U}} \frac{1}{p}\left(I + U + \frac{\bar{a}(1+V')}{s^2 V''}\right) = \frac{1}{p}\left(I + \overline{U}\right).$$

Therefore $\lim_{U \to \overline{U}} j = I + \overline{U} - \lim_{U \to \overline{U}} pR = 0$.

Proof of Proposition 4

Applying the argument of Proposition 3 to the second stage problem, due to (9), we have

$$(r+\bar{a})V^{(2)}(U^{(2)}) + (r+\bar{a})U^{(2)} - \bar{a}p^{(2)}D.$$

Then (A.8) holds in the current case and the proof follows from the proof of Proposition 2 (iii).

Proof of Proposition 5

In the stage i, the draw on the credit line $M^{(i)}$ evolves according to

$$dM_t^{(i)} = rM_t^{(i)}dt + (\bar{a}l \quad r\overline{U})dt \quad j_t^{(i)}dy_t + dC_t,$$

where $(\bar{a}l \quad r\overline{U})dt$ is the coupon payment to the debt holder, dC_t is the compensation to the agent if the credit line is repaid in full, and $j_t^{(i)}dy_t$ is the contribution from the general partner. Then the previous equation shows that agent's continuation utility $U_t = \underline{U} + (\overline{M}^{(i)} \quad M_t^{(i)})$ follows

$$dU_t = rU_t dt \quad \bar{a}l\,dt + j_t dy_t \quad dC_t.$$

If the stage i succeeds at time t, a new credit line is issued with limit $\overline{M}^{(i+1)}$ and the initial balance $\overline{U}^{(i+1)} \quad U^{(i+1)}$. Therefore agent's continuation utility jumps to $U^{(i+1)}$ to start the next stage. Lemma 2 shows that agent's optimal effort is $\bar{a}$. The choice of C, R, and j implies that this capital structure implements the optimal contract.

Appendix B Information acquisition

Denote i as the option exercise time. If it happens before project completion and termination, agent's continuation utility after option exercise depends on the outcome of investigation: If $D = D^l$, the agent gets nothing; if $D = D^h$, the agent gets a new contract with the continuation utility U^{ex} and a milestone bonus R^{ex}. Therefore, agent's expected utility right before the option exercise is $U_i^c = q \; U^{ex} + R^{ex}$. If the project is completed at time n before the investigation and termination, the agent receives reward $R_n^{(1)}$ only when the project is of high type and it succeeds. Therefore, the expected reward right before completion is $p(a_n)qR_n^{(1)}$. Finally, if the project is terminated at t before the completion and the option exercise, the agent receives his outside value $\underline{U}$. Summarizing these previous three sitations, agent's optimization problem at time 0 is

$$\sup_{a \in [0,\bar{a}]} E^a \Big[\int_0^{t\wedge n\wedge i} e^{\;rs} \; dC_s + l(\bar{a} \quad a_s)ds \; + I_{\{n<t\wedge i\}} e^{\;rn} p(a_n)qR_n^{(1)} + I_{\{i<t\wedge n\}} e^{\;ri} U_i^c + I_{\{t<n\wedge i\}} e^{\;rt} \underline{U} \Big]. \tag{B.1}$$

The same argument of Lemma 1 yields that the agent's continuation utility U follows (8) and the agent's optimal effort is

$$a = \begin{cases} \bar{a} & pqR^{(1)} \quad l + U, \\ 0 & \text{otherwise.} \end{cases}$$

Agent's problem after the option exercise is the same as the single-stage situation of Lemma 1. Agent's utility U follows (8) with the initial condition $U_i = U^{ex}$, when $i < n \wedge t$, and the agent's optimal effort after the option exercise is

$$a = \begin{cases} \bar{a} & pR^{(2)} \geq l + U, \\ 0 & \text{otherwise.} \end{cases}$$

Right before the option exercise time, agent's continuation utility satisfies the promise keeping condition

$$U_{i-} = q\left(U^{ex} + R^{ex}\right), \tag{B.2}$$

so that agent's continuation utility right before the option exercise is exactly the expected payoff afterward.

For the principal, the real option's exercise value is introduced in (37) and (38). Solving this constrained optimization problem, we obtain

$$U^{ex}(U) = \min\left\{\frac{U}{q}, \overline{U}^{(2)}\right\} \quad \text{and} \quad R^{ex}(U) = \max\left\{\min\left\{\frac{U}{q} - \overline{U}^{(2)}, \mathrm{D}^h\right\}, 0\right\}, \tag{B.3}$$

where $\overline{U}^{(2)}$ is the maximum continuation utility for problem (36). It is evident from (B.3) that only when U^{ex} reaches $\overline{U}^{(2)}$ does R^{ex} take positive value.

Before the option exercise and project termination, if the project is completed, the principal receives the high type payoff D^h only when the project is of high type and it is successfully completed. Therefore, when the agent exerts his maximum effort, principal's expected payoff, after netting reward to the agent, is $pq(\mathrm{D}^h - R^{(1)})$. Principal's problem at time 0 is

$$\sup_{C, R^{(1)}, i} \mathrm{E}^{\bar{a}}\left[\int_0^{t \wedge n \wedge i} e^{-rs}(-dC_s) + \mathrm{I}_{\{n < t \wedge i\}} e^{-rn} pq\left(\mathrm{D}^h - R\right) + \mathrm{I}_{\{i < t \wedge n\}} e^{-ri} G(U_i) + \mathrm{I}_{\{t < i \wedge n\}} e^{-rt} L\right], \tag{B.4}$$

subject to the incentive compatible constraint $pqR^{(1)} \geq l + U$ and agent's participation constraint.

Using the dynamic programming, we derive the following HJB equation. Principal's value function $V^{(1)}$ satisfies

$$\min\left\{(r+\bar{a})V^{(1)} - \sup_{R^{(1)}}\left\{\bar{a}pq(\mathrm{D}^h - R^{(1)}) + \left((r+\bar{a})U - \bar{a}pqR^{(1)}\right)(V^{(1)})'\right\}, -(V^{(1)})' + 1, V^{(1)} - G\right\} = 0, \tag{B.5}$$

subject to $R \in [0, D^h]$ and $pqR^{(1)} \ge I + U$.

Equation (B.5) indicates that agent's continuation utility U can be in a continuation region where the first group of terms in (B.5) equals to zero, meanwhile $V' > -1$ and $V > G$. It is then suboptimal to pay the agent compensation C or exercise the real option to investigate in this region. Alternatively, there may be a threshold U and it is optimal for the principal to exercise if U reaches U. There is also a threshold $\overline{U}^{(1)} = \inf\{U : (V^{(1)})'(U) = -1\}$. However, because $rU < \bar{a}l$, U decreases along the optimal path and never reaches $\overline{U}^{(1)}$.