

Revenue Maximization in Spectrum Auction for Dynamic Spectrum Access

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Abstract—In this paper, we consider the revenue maximization problem in auctions for dynamic spectrum access. We assume a frequency division method of spectrum sharing with a primary spectrum owner that can divide the available spectrum into sub-bands and sell them to secondary users. We assume that a secondary user’s utility function is linear in the rate it can achieve by using the spectrum. We present an incentive compatible, individual rational and revenue-maximizing mechanism that the spectrum owner can use to divide the spectrum among the strategic (selfish) secondary buyers.

I. INTRODUCTION

Traditional spectrum allocations are done in a static manner where long-term spectrum licenses covering large geographical areas are sold. Under this type of static allocation, there is increasing evidence that spectrum resources are not being efficiently utilized [1]. At the same time, wireless devices are enjoying ever greater capability to detect spectrum availability and flexibility to adjust operating frequencies [2]. These observations have led to a push for dynamic spectrum sharing where the primary spectrum owner may lease spectrum to secondary buyers.

Frequency division multiplexing is one method of spectrum sharing. The primary buyer partitions its available spectrum into sub-bands. Each sub-band is then sold to at most one secondary buyer. Various kinds of auctions have been proposed for the spectrum division problem. In the simplest auction, the available spectrum may be partitioned into n sub-bands (for some finite number n) and a multiple-product auction can be employed. The authors in [3] use a sequential second price auction mechanism where each unit is sequentially allocated using a second-price auction. They study the equilibrium of such an auction and characterizes the accompanying efficiency loss. In [4], the authors consider buyers with strict spectrum demands across multiple channels and find revenue-maximizing auctions.

In contrast to the frequency division method of spectrum sharing, power allocation methods allow different secondary buyers to use the same spectrum. The buyers can distribute power over the available frequency so as to minimize the interference or maximize their rates. Such methods were studied in [5], [6], [7], [8] in a game-theoretic/mechanism design context.

In this paper, we focus on the frequency division method of spectrum sharing. We model the spectrum as a perfectly divisible commodity. The spectrum owner has to find a way

of dividing the spectrum among the secondary buyers and charging payments to the secondary buyers that maximizes its revenue. Since the spectrum owner does not have complete information about the secondary buyers’ utilities, it has to solicit information from them. Based on the information it receives, the spectrum owner decides the distribution of spectrum and payments among the buyers. The spectrum allocation and the payment rule are collectively referred to as the mechanism chosen by the spectrum owner.

We assume that the buyers’ utilities are linear in the expected rate they can achieve from a given amount of spectrum. We further assume that a buyer’s private information is entirely captured by the slope of this linear relation. We interpret this slope as a buyers’ “willingness to pay” for the expected rate it may get. We model the secondary buyers as strategic agents. Thus, once the spectrum owner fixes its allocation and payment rule, a Bayesian game is played among the buyers. The spectrum owner has to find allocation and payment rules that maximize its revenue while ensuring that truth-telling is an equilibrium of the induced Bayesian game among the buyers. The spectrum owner’s problem belongs to the class of Bayesian mechanism design. Bayesian mechanism design is a branch of mathematical economics (see [9], [10], [11], [12] and references therein). Our work is philosophically similar to Myerson’s optimal auction ([13]) of an indivisible good. However, since we assume perfect divisibility of spectrum, our mechanism differs from the mechanism in [13].

Organization of the Paper: The rest of the paper is organized as follows. We formulate the spectrum owner’s optimization problem in Section II. We introduce incentive compatibility and individual rationality as constraints in the primary spectrum owner’s optimization problem. In Section III, we characterize necessary and sufficient conditions for a mechanism to satisfy these constraints. We further provide a candidate solution of the spectrum owner’s problem.

Notation: Set of buyers is denoted by $\mathcal{N} = \{1, 2, \dots, N\}$. For a vector $\theta = (\theta_1, \theta_2, \dots, \theta_N)$, we use θ_{-i} to refer to $(\theta_1, \theta_2, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_N)$. We use the symbol \mathbb{E} for expectation operator. The subscript used with \mathbb{E} denotes the random variables with respect to which the expectation is taken.

II. PROBLEM FORMULATION

We consider a spectrum market with a seller that owns W Hz of bandwidth and N potential buyers (buyers). We

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assume a frequency division multiplexing model for spectrum sharing, that is, the seller can divide the available spectrum into different sub-bands and allocate them to different buyers. We assume the spectrum is a perfectly divisible commodity and the size of sub-band for each buyer is decided by the seller. We now explain various components of our model in detail:

- 1) *The buyers:* We assume each buyer is a distinct transmitter-receiver pair that can communicate over a channel with Gaussian noise. If buyer i receives x Hz of bandwidth and its channel gain is h_{ii} , then it gets a rate given as:

$$R(x) = x \log \left(1 + \frac{h_{ii}P}{N_0x} \right) \quad (1)$$

We assume that at the time of spectrum transaction, buyers and the seller only have probabilistic information about the channel gains. That is, for each buyer i , the channel gain h_{ii} is a random variable with density (or PMF) g_i . The density g_i is common knowledge among the buyers and the seller. Thus, if buyer i receives x Hz of bandwidth, its *expected* rate is given as

$$\psi_i(x) = \int x \log \left(1 + \frac{h_{ii}P}{N_0x} \right) g_i(h_{ii}) dh_{ii} \quad (2)$$

We assume that the integral in (2) is well-defined for all $0 \leq x \leq W$.

Further, we assume that a buyer's utility is characterized by a single real number θ_i . We call θ_i buyer i 's *type*. If buyer i has type θ_i , its utility from getting x Hz of bandwidth and paying t amount of money is given as:

$$u_i(x, t, \theta_i) = \theta_i \psi_i(x) - t \quad (3)$$

In other words, a buyer's utility is linear in the expected rate and the monetary payment. We can interpret θ_i as buyer i 's "willingness to pay" - it is the maximum price per unit of expected rate that the buyer is willing to pay. We assume that θ_i , $i \in \mathcal{N}$ are independent random variables; We assume that for each buyer i , θ_i is private information, that is, only buyer i knows the true value of its type; We assume that $\theta_i \in \Theta_i := [\theta_i^{min}, \theta_i^{max}]$ and the sets Θ_i are common knowledge. All buyers other than buyer i and the seller have a prior probability density function $f_i(\cdot)$ (with the corresponding CDF being $F_i(\cdot)$) on θ_i ; we assume that these densities are common knowledge. We define $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ and $\Theta := \times_{i=1}^N [\theta_i^{min}, \theta_i^{max}]$.

- 2) *The Seller:* We assume that seller knows the distributions g_i of each buyers channel gain and the distributions f_i of each buyer's type. We assume that the seller's utility is the total money he gets from the buyers.
- 3) *The Mechanism:* The seller asks each buyer to report his type. On receiving the reports from all buyers, the seller uses an allocation rule $q = (q_1, q_2, \dots, q_N)$ and payment

function $t = (t_1, t_2, \dots, t_N)$,

$$q_i : \Theta \rightarrow [0, W] \quad \text{for } i = 1, 2, \dots, N, \quad (4)$$

$$t_i : \Theta \rightarrow R_+ \quad \text{for } i = 1, 2, \dots, N, \quad (5)$$

with $\sum_{i=1}^N q_i(\theta) \leq W$, where $q_i(\theta)$ is the amount of spectrum given to buyer i and $t_i(\theta)$ is the payment charged to buyer i when the type vector reported is θ .

Once the mechanism (q, t) has been announced, it induces a Bayesian game among the buyers. Each buyer observes his own type but has only a probability distribution on other players' types. A buyer can report any type (not necessarily its true type) if it expects a higher utility by mis-reporting.

A. Incentive Compatibility and Individual Rationality

We define the following properties for a mechanism.

- 1) *Incentive Compatibility:* A mechanism (q, t) is said to be incentive compatible if for each $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, we have

$$\begin{aligned} & \mathbb{E}_{\theta_{-i}} [\theta_i \psi_i(q_i(\theta)) - t_i(\theta)] \\ & \geq \mathbb{E}_{\theta_{-i}} [\theta_i \psi_i(q_i(r_i, \theta_{-i})) - t_i(r_i, \theta_{-i})] \quad \forall r_i \in \Theta_i. \end{aligned} \quad (6)$$

Incentive compatibility guarantees that truthful reporting is a Bayesian Nash equilibrium for the game induced by the mechanism. That is, each buyer prefers truthful reporting to any other strategy given that all other buyers are truthful.

- 2) *Individual Rationality:* A mechanism (q, t) is said to be individually rational if for each $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, we have

$$\mathbb{E}_{\theta_{-i}} [\theta_i \psi_i(q_i(\theta)) - t_i(\theta)] \geq 0. \quad (7)$$

Individual rationality guarantees that at the truthful Bayesian Nash equilibrium, each buyer has a utility no less than that obtained by not participating in the spectrum allocation process at all.

In our search for finding the revenue-maximizing mechanism, we will restrict to the class of mechanisms that are incentive compatible and individual rational. Revelation principle for Bayesian mechanism design ([14]) ensures that any spectrum allocation and payments achieved at an equilibrium of a Bayesian game of any mechanism can be achieved by an incentive compatible mechanism. Thus, restricting to incentive compatible mechanism incurs no loss of revenue. We impose individual rationality as a natural requirement for a mechanism that induces players to participate in the mechanism.

B. Revenue Maximization

We have the following problem for the seller

Problem 1: The seller's optimization problem is to choose a feasible mechanism (q, t) that satisfies equations (6) and (7) and maximizes his expected revenue given as:

$$\mathbb{E}_{\theta} \left\{ \sum_{i=1}^N t_i(\theta) \right\}$$

III. ANALYSIS

We start with the following lemma for the function ψ defined in (2).

Lemma 1: The function $\psi(x)$ is non-decreasing and concave in x .

Proof: See Appendix A ■

A. Characterizing Incentive Compatibility and Individual Rationality

In this Section, we derive necessary and sufficient conditions for a mechanism to be incentive compatible and individually rational. Let (q, t) be any mechanism selected by the seller. In order to characterize incentive compatibility and individual rationality for buyer i , we will adopt buyer i 's perspective. Let θ_i be the type of buyer i . Buyer i knows his own type. However, when the seller asks the buyer to report his type, he may report any type r_i between θ_i^{min} and θ_i^{max} . We define the following functions:

Definition 1: 1) Given a mechanism (q, t) , we define for each $r_i \in \Theta_i$,

$$Q_i(r_i) := \mathbb{E}_{\theta_{-i}}[\psi_i(q_i(r_i, \theta_{-i}))] \quad (8)$$

$Q_i(r_i)$ is the expected rate under the given mechanism that user i will get if he reports r_i while all other users report truthfully. Note that the expectation is over the type of all other users θ_{-i} . Similarly, the expected payment that user i will pay is given as

$$T_i(r_i) := \mathbb{E}_{\theta_{-i}}[t_i(r_i, \theta_{-i})], \quad (9)$$

2) Given a mechanism (q, t) , we define for each $\theta_i, r_i \in \Theta_i$, we define

$$U_i(\theta_i, r_i) = \theta_i Q_i(r_i) - T_i(r_i) \quad (10)$$

$U_i(\theta_i, r_i)$ is the expected utility for user i if its type is θ_i and it reports r_i . Once again, the expectation is over the type of all other users θ_{-i} .

We can re-write the incentive compatibility and individual rationality constraints for user i in terms of the functions defined above.

Incentive Compatibility for buyer i :

$$\begin{aligned} U_i(\theta_i, \theta_i) &\geq U_i(\theta_i, r_i), \quad \theta_i, r_i \in \Theta_i \\ \iff \theta_i Q_i(\theta_i) - T_i(\theta_i) &\geq \theta_i Q_i(r_i) - T_i(r_i), \quad \theta_i, r_i \in \Theta_i \end{aligned}$$

Individual Rationality for buyer i :

$$\begin{aligned} U_i(\theta_i, \theta_i) &\geq 0, \quad \theta_i \in \Theta_i \\ \iff \theta_i Q_i(\theta_i) - T_i(\theta_i) &\geq 0, \quad \theta_i \in \Theta_i \end{aligned}$$

We can now characterize incentive compatibility and individual rationality by the following theorem.

Theorem 1: A mechanism (q, t) is incentive compatible and individually rational if and only if $Q_i(r_i)$ is non-decreasing in r_i and

$$T_i(r_i) = K_i + r_i Q_i(r_i) - \int_{\theta_i^{min}}^{r_i} Q_i(s) ds, \quad (11)$$

where $K_i = (T_i(\theta_i^{min}) - \theta_i^{min} Q_i(\theta_i^{min})) \leq 0$.

Proof: See Appendix B ■

B. Seller's Optimization Problem

The seller's objective can be written as:

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}_{\theta} \{t_i(\theta)\} &= \sum_{i=1}^N \mathbb{E}_{\theta_i} [\mathbb{E}_{\theta_{-i}} t(\theta_i, \theta_{-i})] \\ &= \sum_{i=1}^N \mathbb{E}_{\theta_i} [T_i(\theta_i)] \end{aligned} \quad (12)$$

Further, because of Theorem 1, we can write each term in the summation in (12) as

$$\begin{aligned} \mathbb{E}_{\theta_i} [T_i(\theta_i)] &= K_i \\ &+ \int_{\theta_i^{min}}^{\theta_i^{max}} \left[\theta_i Q_i(\theta_i) - \int_{\theta_i^{min}}^{\theta_i} Q_i(s) ds \right] f_i(\theta_i) d\theta_i \end{aligned} \quad (13)$$

The integral in (13) can be written as:

$$\begin{aligned} &\int_{\theta_i^{min}}^{\theta_i^{max}} \theta_i Q_i(\theta_i) f_i(\theta_i) d\theta_i - \int_{\theta_i^{min}}^{\theta_i^{max}} \int_{\theta_i^{min}}^{\theta_i} Q_i(s) ds f_i(\theta_i) d\theta_i \\ &= \int_{\theta_i^{min}}^{\theta_i^{max}} \theta_i Q_i(\theta_i) f_i(\theta_i) d\theta_i - \int_{\theta_i^{min}}^{\theta_i^{max}} Q_i(s) \int_s^{\theta_i^{max}} f_i(\theta_i) d\theta_i ds \\ &= \int_{\theta_i^{min}}^{\theta_i^{max}} \theta_i Q_i(\theta_i) f_i(\theta_i) d\theta_i - \int_{\theta_i^{min}}^{\theta_i^{max}} Q_i(s) (1 - F_i(s)) ds \end{aligned} \quad (14)$$

Using the definition of $Q_i(\cdot)$ from (8) in (14), we get

$$\begin{aligned} &= \int_{\theta} \theta_i \psi_i(q_i(\theta)) f(\theta) d\theta \\ &- \int_{\theta_{-i}} \int_{\theta_i^{min}}^{\theta_i^{max}} \psi_i(q_i(s, \theta_{-i})) (1 - F_i(s)) f_{-i}(\theta_{-i}) d\theta_{-i} \\ &= \int_{\theta} \left[\psi_i(q_i(\theta)) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \right] f(\theta) d\theta \end{aligned} \quad (15)$$

In the economics literature the term $\left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right)$ appearing in the integral in (15) is called *virtual type*.

Using (12), (13) and (15), we can write the total expected revenue as:

$$\begin{aligned} &\sum_{i=1}^N K_i \\ &+ \sum_{i=1}^N \int_{\theta} \left[\psi_i(q_i(\theta)) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \right] f(\theta) d\theta \end{aligned} \quad (16)$$

A mechanism (q, t) for which $K_i = 0$, $i \in \mathcal{N}$ and which maximizes

$$\begin{aligned} &\sum_{i=1}^N \int_{\theta} \left[\psi_i(q_i(\theta)) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \right] f(\theta) d\theta \\ &= \int_{\theta} \sum_{i=1}^N \left[\psi_i(q_i(\theta)) \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \right] f(\theta) d\theta \end{aligned} \quad (17)$$

while satisfying $\sum_{i=1}^N q(\theta) \leq W$ as well as the conditions of Theorem 1 will be a revenue-maximizing, incentive compatible and individually rational mechanism.

C. Regularity Condition and A Candidate Solution

We impose the following assumption on the virtual type of each buyer which is often called *regularity condition*.

Assumption: For each buyer i , $\left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right)$ is increasing in θ_i .

We can now propose a candidate solution for the seller.

Theorem 2: For each $\theta \in \Theta$, let $q_i(\theta)$, $i = 1, 2, \dots, N$ be the solution of the following optimization problem:

$$\begin{aligned} \arg \max_x \quad & \sum_{i=1}^N \left\{ \psi_i(x_i) \left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} \right) \right\} \\ \text{subject to} \quad & \sum_{i=1}^N x_i \leq W. \end{aligned} \quad (18)$$

and let $t_i(\theta)$, $i = 1, 2, \dots, N$ be given as:

$$t_i(\theta) = \theta_i \psi_i(q_i(\theta)) - \int_{\theta_i^{min}}^{\theta_i} \psi_i(q_i(s, \theta_{-i})) ds. \quad (19)$$

Then, (q, t) is an incentive compatible and individually rational mechanism that maximizes the seller's expected revenue.

Proof: By definition, $q_i(\theta)$, $i = 1, 2, \dots, N$ achieves the maximum value of $\sum_{i=1}^N \left[\psi_i(q_i(\theta)) \left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} \right) \right]$ for each θ . Hence it maximizes the integral in (17).

We will now show that (q, t) satisfies the characterization of incentive compatibility and individual rationality in Theorem 1 with $K_i = 0$.

$$\begin{aligned} T_i(r_i) &= \mathbb{E}_{\theta_{-i}} [t_i(r_i, \theta_{-i})] \\ &= \theta_i \int_{\theta_{-i}} \psi_i(q_i(r_i, \theta_{-i})) f_{-i}(\theta_{-i}) d\theta_{-i} \\ &\quad - \int_{\theta_{-i}} \int_{\theta_i^{min}}^{\theta_i} \psi_i(q_i(s, \theta_{-i})) ds f_{-i}(\theta_{-i}) d\theta_{-i} \\ &= \theta_i Q_i(r_i) - \int_{\theta_i^{min}}^{\theta_i} Q_i(s) ds \end{aligned} \quad (20)$$

Thus, $T_i(\cdot)$ satisfies (11) of Theorem 1 with $K_i = 0$.

We will now show that for each θ_{-i} , $\psi_i(q_i(\theta_i, \theta_{-i}))$ is non-decreasing in θ_i . This, when averaged over θ_{-i} , will imply monotonicity of $Q_i(\cdot)$.

Consider any value of θ_{-i} . Let $w_i(\theta_i) := \left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right)$. By assumption, $w_i(\theta_i)$ is increasing in θ_i . Let $a, b \in \Theta_i$ with $a < b$. Let $(x_1^a, x_2^a, \dots, x_N^a)$ and $(x_1^b, x_2^b, \dots, x_N^b)$ be the solutions for the optimization problem (18) with $\theta_i = a$ and b respectively. Then, we must have,

$$\begin{aligned} & \psi_i(x_i^a) w_i(a) + \sum_{j \neq i} \psi_j(x_j^a) w_j(\theta_j) \\ & \geq \psi_i(x_i^b) w_i(a) + \sum_{j \neq i} \psi_j(x_j^b) w_j(\theta_j) \end{aligned} \quad (21)$$

Similarly,

$$\begin{aligned} & \psi_i(x_i^b) w_i(b) + \sum_{j \neq i} \psi_j(x_j^b) w_j(\theta_j) \\ & \geq \psi_i(x_i^a) w_i(b) + \sum_{j \neq i} \psi_j(x_j^a) w_j(\theta_j) \end{aligned} \quad (22)$$

Summing (21) and (22) gives

$$\psi_i(x_i^b) (w_i(b) - w_i(a)) \geq \psi_i(x_i^a) (w_i(b) - w_i(a)) \quad (23)$$

Since, $(w_i(b) - w_i(a)) > 0$, (23) implies $\psi_i(x_i^b) \geq \psi_i(x_i^a)$. This establishes the monotonicity of $\psi_i(q_i(\theta_i, \theta_{-i}))$ in θ_i . ■

Theorem 2 thus identifies a mechanism that solves the seller's optimization problem. Note that for each θ , finding the allocated spectrum for each user involves solving the optimization problem in (18). We now show that this optimization is a convex optimization problem.

Lemma 2: The optimization problem in (18) is a convex-optimization problem.

Proof: We know from Lemma 1 that $\psi_i(x_i)$ is a concave function of x_i . However, the objective in the (18) may not be concave since for some i , $\psi_i(x_i)$ may be weighted by a negative multiplier $w_i(\theta_i) := \left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right)$. If this multiplier is negative, then the objective function is maximized by choosing $x_i = 0$ since $\psi_i(0) = 0$. Thus, the objective function in (18) can be replaced by

$$\sum_{i: w_i(\theta_i) \geq 0} \psi_i(x_i) w_i(\theta_i)$$

This is now a concave function of x_i , $i = 1, 2, \dots, N$. Hence, the maximization in (18) is equivalent to a convex optimization problem. ■

D. Interpretation/Discussion of the Mechanism

An omniscient seller who knew buyers' type could have charged each buyer the maximum price it was willing to pay. Thus, given an allocation rule q , an omniscient observer could have obtained a tax amount equal to $\theta_i \psi_i(q_i(\theta))$ from user i when the type realization was θ . Our less informed seller, however, has to provide a subsidy of $\int_{\theta_i^{min}}^{\theta_i} \psi_i(q_i(s, \theta_{-i})) ds$ to buyer i , $i \in \mathcal{N}$ to ensure that buyer i reveals its true type.

The tax paid by a user can be more intuitively explained using the following function:

$$Z_i(y, \theta_{-i}) := \inf \{s \in \Theta_i | \psi_i(q_i(s, \theta_{-i})) \geq y\}$$

Thus, $Z_i(y, \theta_{-i})$ is the minimum willingness to pay that buyer i should report in order to get at least y amount of rate when other buyers type is θ_{-i} . We also define the bandwidth that buyer i will obtain by reporting his type to be θ_i^{min} as

$$q_i^{min}(\theta_{-i}) := q_i(\theta_i^{min}, \theta_{-i})$$

Note that if $\psi_i(q_i(s, \theta_{-i}))$ is a one to one function of s , then for y in the range of this function, $Z_i(y, \theta_{-i}) = s$ if and only if $\psi_i(q_i(s, \theta_{-i})) = y$. The tax function for user i is given as:

$$t_i(\theta) = \theta_i \psi_i(q_i(\theta)) - \int_{\theta_i^{min}}^{\theta_i} \psi_i(q_i(s, \theta_{-i})) ds \quad (24)$$

Figure 1 shows the variation of buyer i 's expected rate as a

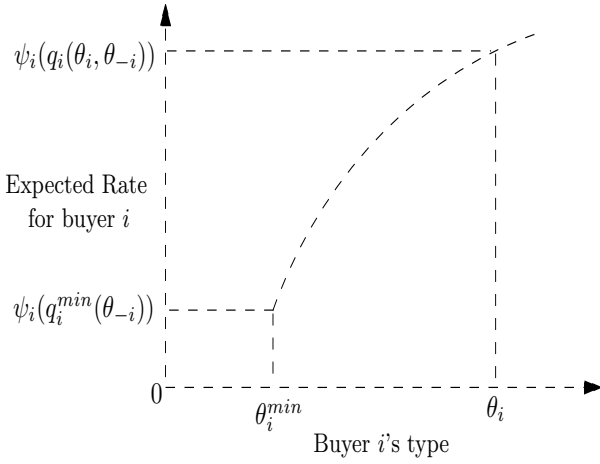


Fig. 1. Expected rate for buyer i vs. buyer i 's type for a fixed θ_{-i}

function of its type for a given θ_{-i} . The tax paid by user i is equal to the area bounded by vertical lines at 0 and θ_i and horizontal lines at 0 and $\psi_i(q_i(\theta_i, \theta_{-i}))$ minus the area under the expected rate-type curve. An alternative evaluation of this area can be obtained by the following expression:

$$t_i(\theta) = \theta_i^{min} \psi_i(q_i^{min}(\theta_{-i})) + \int_{\psi_i(q_i^{min}(\theta_{-i}))}^{\psi_i(q_i(\theta))} Z_i(y, \theta_{-i}) dy. \quad (25)$$

Thus, each buyer pays a base amount of $\theta_i^{min} \psi_i(q_i^{min}(\theta_{-i}))$. In addition, for each infinitesimal increment in rate from y to $y + dy$, the buyer is charged the minimum price that would obtain the rate y when other buyers have types θ_{-i} .

E. Computational Aspects

On receiving the types from the buyers, the seller needs to solve a convex optimization problem to find the optimal allocations according to the mechanism in Theorem 2. Efficient computational methods are well-known for such computational problems. The computational bottleneck in the mechanism of Theorem 2 comes from the tax equation. In order to evaluate the tax for the buyer i , the seller needs to evaluate the integral $\int_{\theta_i^{min}}^{\theta_i} \psi_i(q_i(s, \theta_{-i})) ds$. To evaluate the integral, we need to know the allocation $q_i(s, \theta_{-i})$ for all $\theta_i^{min} \leq s \leq \theta_i$. Thus, the seller has to solve a series of convex optimization problems. In practice, the integral may be approximated by a Riemann sum, so that the seller has to solve a finite number of optimization problems.

A consequence of approximating the tax function is that the seller can only guarantee *approximate incentive compatibility* and *approximate individual rationality*. In other words, if the seller calculates an under-approximation of the tax to within ϵ of the correct value, it can guarantee that users cannot increase their utility by more than ϵ if they misreport their type or choose not to participate in the spectrum allocation process.

IV. CONCLUSIONS

We derived a revenue maximizing mechanism for a spectrum owner that can divide its spectrum into sub-bands and lease them to secondary users for a fee. We assumed a frequency division method of spectrum sharing, that is, different secondary users use different parts of the spectrum and do not interfere with each other. We derived our results under the assumptions that users' types are independent random variables with densities that are common knowledge among the users and the seller. We assumed that only user i observes his type. We also assumed that all buyers and the seller have probabilistic information about the channel gains.

The linear relationship between a buyer's utility and the expected rate it can achieve is a critical assumption of our analysis. This allowed us to completely characterize a buyer's private information by a single parameter θ_i . The characterization of incentive compatible and individually rational mechanism obtained in Theorem 1 is critically dependent on the uni-dimensionality of each buyer's private information as captured by its type θ_i . Revenue maximizing mechanisms with general models of buyers' utilities and multi-dimensional private information remain an open problem.

APPENDIX A PROOF OF LEMMA 1

$$\psi'(x) = \int \left[\log\left(1 + \frac{h_{ii}P}{N_0x}\right) - \frac{h_{ii}P}{N_0x + h_{ii}P} \right] g(h_{ii}) dh_{ii}, \quad (26)$$

$$\psi''(x) = \int \left[-\frac{h_{ii}P}{(h_{ii}P + N_0x)x} + \frac{h_{ii}PN_0}{(h_{ii}P + N_0x)^2} \right] g(h_{ii}) dh_{ii} \quad (27)$$

$$= \int \left[-\frac{(h_{ii}P)^2}{(h_{ii}P + N_0x)^2 x} \right] g(h_{ii}) dh_{ii} \leq 0 \quad (28)$$

Equation (28) establishes the concavity of $\psi(x)$. Further, by (26), $\psi'_i(0) = +\infty$ and $\lim_{x \rightarrow \infty} \psi'(x) = 0$. This combined with the fact that $\psi'(x)$ is a non-increasing function (because of $\psi''(x) \leq 0$), implies that $\psi'(x) > 0$, for $x \geq 0$. Thus, $\psi(x)$ is a non-decreasing function of x .

APPENDIX B PROOF OF THEOREM 1

Sufficiency: First assume that (q, t) is a mechanism for which $Q_i(r_i)$ is non-decreasing in r_i and equation (11) is true. We will show that (q, t) is incentive compatible and individually rational for buyer i . For any $\theta_i \in \Theta_i$, we have

$$U_i(\theta_i, \theta_i) = \theta_i Q_i(\theta_i) - T_i(\theta_i) = \int_{\theta_i^{min}}^{\theta_i} Q_i(s) ds - K \quad (29)$$

$$\geq 0, \quad (30)$$

where we used (11) in (29) and the non-negativity of Q_i and of $(-K_i)$ in (30). Thus, (q, t) is individually rational for buyer i . Further,

$$\begin{aligned} & U_i(\theta_i, \theta_i) - U_i(\theta_i, r_i) \\ &= \int_{\theta_i^{min}}^{\theta_i} Q_i(s) ds - \theta_i Q(r_i) + r_i Q(r_i) \\ & \quad - \int_{\theta_i^{min}}^{r_i} Q_i(s) ds \end{aligned} \quad (31)$$

Consider the case when $r_i < \theta_i$. Then, the right hand side of (31) can be written as

$$\begin{aligned} & \int_{r_i}^{\theta_i} Q_i(s) ds - (\theta_i - r_i) Q(r_i) \\ & \geq 0, \end{aligned} \quad (32)$$

where we used the non-decreasing nature of Q_i in (32). Similarly, if $r_i > \theta_i$, the right hand side of (31) can be written as

$$\begin{aligned} & - \int_{\theta_i}^{r_i} Q_i(s) ds + (r_i - \theta_i) Q(r_i) \\ & \geq 0, \end{aligned} \quad (33)$$

which again follows from the non-decreasing nature of Q_i . Thus, we have that

$$U_i(\theta_i, \theta_i) \geq U_i(\theta_i, r_i),$$

for all $\theta_i, r_i \in \Theta_i$, which establishes incentive compatibility for buyer i .

Necessity: Let (q, t) be an incentive compatible and individually rational mechanism. Let $a, b \in \Theta_i$ with $a < b$. Incentive compatibility implies that:

$$aQ_i(a) - T_i(a) \geq aQ_i(b) - T_i(b) \quad (34)$$

and

$$bQ_i(b) - T_i(b) \geq bQ_i(a) - T_i(a) \quad (35)$$

Adding (34) and (35) gives

$$Q_i(b)(b - a) \geq Q_i(a)(b - a) \quad (36)$$

Since $(b - a) > 0$, we must have $Q_i(b) \geq Q_i(a)$ -which establishes monotonicity of Q_i .

We define $V_i(\theta_i) := U_i(\theta_i, \theta_i)$. That is, $V_i(\theta_i)$ is the expected utility of buyer i with type θ_i under truthful reporting. Because of incentive compatibility, we have

$$\begin{aligned} V_i(\theta_i) &= \max_{r_i \in \Theta_i} U_i(\theta_i, r_i) \\ &= \max_{r_i \in \Theta_i} \theta_i Q_i(r_i) - T_i(r_i), \end{aligned}$$

which implies that $V_i(\theta_i)$ is the maximum of a family of affine functions of θ_i . Thus, $V_i(\theta_i)$ is a convex function and is differentiable everywhere except for at most countably many points.

Consider the following limit

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{V_i(\theta_i + \delta) - V_i(\theta_i)}{\delta} \\ & \geq \lim_{\delta \rightarrow 0} \frac{U_i(\theta_i + \delta, \theta_i) - V_i(\theta_i)}{\delta} \\ & = \lim_{\delta \rightarrow 0} \frac{(\theta_i + \delta)Q_i(\theta_i) - T_i(\theta_i) - \theta_i Q_i(\theta_i) + T_i(\theta_i)}{\delta} \\ & = Q_i(\theta_i) \end{aligned} \quad (37)$$

Similarly, we have

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{V_i(\theta_i) - V_i(\theta_i - \delta)}{\delta} \\ & \leq \lim_{\delta \rightarrow 0} \frac{V_i(\theta_i) - U_i(\theta_i - \delta, \theta_i)}{\delta} \\ & = \lim_{\delta \rightarrow 0} \frac{\theta_i Q_i(\theta_i) - T_i(\theta_i) - (\theta_i - \delta)Q_i(\theta_i) + T_i(\theta_i)}{\delta} \\ & = Q_i(\theta_i) \end{aligned} \quad (38)$$

Equations (37) and (38) imply that $V_i'(\theta_i) = Q_i(\theta_i)$. Thus, for any $r_i \in \Theta_i$,

$$\begin{aligned} V_i(r_i) &= V_i(\theta_i^{min}) + \int_{\theta_i^{min}}^{r_i} Q_i(s) ds \\ & \implies r_i Q_i(r_i) - T_i(r_i) = \theta_i^{min} Q_i(\theta_i^{min}) - T_i(\theta_i^{min}) \\ & \quad + \int_{\theta_i^{min}}^{r_i} Q_i(s) ds \end{aligned} \quad (39)$$

Rearranging (39) gives

$$\begin{aligned} T_i(r_i) &= (T_i(\theta_i^{min}) - \theta_i^{min} Q_i(\theta_i^{min})) \\ & \quad + r_i Q_i(r_i) - \int_{\theta_i^{min}}^{r_i} Q_i(s) ds \end{aligned} \quad (40)$$

Defining $K_i = (T_i(\theta_i^{min}) - \theta_i^{min} Q_i(\theta_i^{min}))$, we get (11) of Theorem 1 from (40). Note that individual rationality at θ_i^{min} implies that

$$\theta_i^{min} Q_i(\theta_i^{min}) - T_i(\theta_i^{min}) \geq 0,$$

which implies that $K_i \leq 0$.

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